Dynamic Analysis of Intermittent-Motion Mechanisms Through the Combined Use of Gauss Principle and Logical Functions

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1 Introduction

Intermittent-motion mechanisms play an important role in modern technology. For instance they are key elements of many automatic machines. Scientific literature records different modelling analyses of this kind of mechanism (e.g. [1, 2]). Due to the widespread use of such devices, their analysis and design taking into account impact phenomena appears to be significant.

The dynamic simulation of intermittent motion involves several issues. For example the presence of impact and sudden changes in velocity leads to the requirement of including transient mechanics. Several mathematical models of impact could be found in most of the investigations which deal with mechanism clearances. However in ratchet mechanisms the impact is an usual event whose presence does not depend on the clearances.

The main contributions presented in this paper are:

- an extension of the dynamic formulation proposed by Udwadia and Kalaba [3] to the analysis of impacts;
- an engineering model for the analysis of the impact phenomena in ratchet mechanisms;
- a procedure of optimal design of ratchet mechanism.

2 Brief summary of the Gauss-Udwadia-Kalaba dynamic formulation

The main advantages of this formulation concern with the possibility of reducing the equations of motion to a system of ordinary differential equations (ODE), even in presence of redundant constraints or sudden topology changes. The numerical efficiency of this formulation has been discussed in the following bibliographical references [4, 7].

Let us denote with

- $\{F\}$ the vector of external generalized forces;
- $[M]$ the mass matrix;
- $[\Psi_q]$ the Jacobian of constraints equations;
- $\{q\}$ the vector of generalized coordinates;
- $\{\gamma\} = -\left([\Psi_q]\{\dot{q}\}\right)_q \{\dot{q}\} - 2[\Psi_{qt}]\{\dot{q}\} - \{\Psi_{tt}\}$;
- the upperscript $+$ denotes the operation of pseudoinverse of a matrix.

When a redundant set of coordinates is used, the following system of differential-algebraic system of equations (DAE) is obtained

$$\begin{bmatrix} M & \Psi_q^T \\ \Psi_q & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} F \\ \gamma \end{Bmatrix} \tag{1}$$

If we let

$$[M]^{-1} = [M]^{-\frac{1}{2}} [M]^{-\frac{1}{2}} \ , \quad [D] = [\Psi_q] [M]^{-\frac{1}{2}} \ , \quad \{\ddot{q}_f\} = [M]^{-1}\{F\} \tag{2}$$

one can demonstrate [3, 4, 7] that

$$\{\ddot{q}\} = \{\ddot{q}_f\} + [M]^{-\frac{1}{2}} [D]^+ \left(\{\gamma\} - [\Psi_q]\{\ddot{q}_f\}\right) \ . \tag{3}$$

3 Multibody dynamics simulation with impact

Let us assume that the impact is at time $t = t_I \in [t_1 \ t_2]$. The variation of speed $\{\Delta\dot{q}\} = \left\{\dot{q}\left(t_I^+\right)\right\} - \left\{\dot{q}\left(t_I^-\right)\right\}$ is computed by solving the system

Fig. 1. Impact between bodies

$$\begin{bmatrix} M & \Psi_q^T \left\{\frac{\partial s}{\partial\{q\}}\right\}^T \\ \Psi_q & 0 & 0 \\ \left\{\frac{\partial s}{\partial\{q\}}\right\} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\dot{q} \\ \lambda_I \\ -p \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -(1+e)\left\{\frac{\partial s}{\partial\{q\}}\right\}\left\{\dot{q}\left(t_I^-\right)\right\} \end{Bmatrix} \ . \tag{4}$$

where, as shown in Figure 1, s is the distance between the impacting bodies, e the coefficient of restitution. System (4) can be solved efficiently taking into account the partitioned nature of its coefficient matrix

$$[A] = \left[\begin{array}{c:c} A_{11} & A_{12} \\ \hdashline A_{21} & A_{22} \end{array}\right] \ , \tag{5}$$

where $[A_{11}] = [M]$, $[A_{12}] = \left[\Psi_q^T \, \vdots \, \left\{ \dfrac{\partial s}{\partial \{q\}} \right\}^T \right]$, $[A_{22}] = [0]$, $[A_{21}] = [A_{12}]^T$.

Through inversion one obtains

$$[A]^{-1} = \left[\begin{array}{cc} M^{-1} - M^{-1}A_{12}\left(A_{21}M^{-1}A_{12}\right)^{-1}A_{21}M^{-1} & M^{-1}A_{12}\left(A_{21}M^{-1}A_{12}\right)^{-1} \\ \left(A_{21}M^{-1}A_{12}\right)^{-1}A_{21}M^{-1} & -\left(A_{21}M^{-1}A_{12}\right)^{-1} \end{array} \right] .$$

Therefore, from system (4), after the introduction of the equalities

$$\{v\} = \left\{ 0 \, \vdots \, -(1+e)\left\{ \frac{\partial s}{\partial \{q\}} \right\} \left\{ \dot{q}\left(t_I^-\right) \right\} \right\}^T \ , \tag{6}$$

and $[D_1] = [A_{21}]\,[M]^{-\frac{1}{2}}$ follows

$$\{\Delta\dot{q}\} = [M]^{-\frac{1}{2}}\,[D_1]^+\,\{v\} \ . \tag{7}$$

By means of this original equation, the velocity jump $\{\Delta\dot{q}\}$ due to impact is computed. It shares the features of the dynamic formulation expressed by (3). In particular, (7) can be effectively used for the case of mechanisms with redundant constraints and varying kinematic structure, as in the case of intermittent motion mechanisms.

Fig. 2. Flow-chart of the computational procedure for the analysis of intermittent mechanisms

The flow chart shown in Figure 2 summarizes the main steps of the procedure to be adopted in the case of mechanisms with varying kinematic structure. The logical functions are used to monitor such variation.

4 A method for computing stress due to impact

In this section a methodology for computing stress due to anelastic impact is presented. According to W. Goldsmith [5] the transverse collision of a rigid striker with a beam could be treated as an equivalent mass–spring–damper system. The dynamic deflection curve $w(x,t)$ of the beam is assumed geometrically similar to the modal shape $u(x)$, therefore

$$w(x,t) = u(x)\,\eta(t) \ . \tag{8}$$

The values of the equivalent beam mass m and stiffness k follow from the well known conditions

$$\frac{1}{2}m\dot{\eta}^2 = \frac{1}{2}\rho\int_0^l A(x)\left(\frac{\partial w}{\partial t}\right)^2 dx\ , \qquad \frac{1}{2}k\eta^2 = \frac{1}{2}E\int_0^l I_n(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx\ , \tag{9}$$

where $A(x)$ is the area of the cross section, $I_n(x)$ the moment of inertia, E the Young modulus and ρ the mass density and x coordinate along the beam axis.

The differential equation of motion of the equivalent system in terms of displacement $\eta(t)$ is

$$m\ddot{\eta}(t) + c\dot{\eta}(t) + k\eta(t) = 0\ , \tag{10}$$

with $c = 2\zeta\sqrt{km}$ damping coefficient. The initial conditions to be imposed are $\eta(0) = 0$ and $\dot{\eta}(0) = v_I$. These represent the kinematic conditions of the beam at impact time. In fact the position instantaneously is not changed while a sudden variation of its velocity is registered. The value of initial velocity v_I follows from (7).

For the complete definition of the dynamic deflection curve, as reported in (8), the modal shape curve $u(x)$ needs to be evaluated. The Rayleigh-Ritz method has been applied to compute the first and second natural frequencies and their relative modal shapes. For this purpose a static deflection curve, consistent with boundary conditions of the beam, is expressed in polynomial form:

$$u(x) = \sum_{i=0}^{n} a_i x^i\ . \tag{11}$$

The coefficients a_i $(i = 0,\ldots,n)$ are computed solving the linear system of equations

$$\frac{\partial V}{\partial a_i} - \Lambda^2\frac{\partial T}{\partial a_i} = 0\ (i = 0\ldots n)\ . \tag{12}$$

where V and T represent the potential and kinetic energy of the beam, respectively.

Finally, the time varying expression of the bending moment M_b and the related bending stress σ_b along the beam are computed, respectively, by means of the following equations

$$M_b = -EI_n(x)\frac{\partial^2 w}{\partial x^2}\ , \qquad \sigma_b = \frac{M_b}{W_b}\ . \tag{13}$$

where W_b is the section modulus.

The Figure 3 reports the flow-chart summarizing the steps of the stress recovery methodology used for the optimization of the ratchet mechanism.

Fig. 3. Beam stress recovery method flow-chart

5 Optimal design of a ratchet mechanism

Ratchets belong to the class of intermittent motion mechanisms. Although the component bodies are subjected to impact, their design is usually based on the hypothesis of static application of loads [6]. Thus, it seems appropriate to apply the previously discussed theory to the optimal design of the toothed wheel. The main geometry of the system is depicted in Figure 4.

Fig. 4. Nomenclature

The dimensions and the inertial properties of the links are summarized in Table 1[1] and Table 2. The initial configuration of the system is set at $\phi = 77°$,

Table 1. Geometric dimensions (See Figure 4)

$d = 0.130$ m	Distance between the fixed hinges
$a = 0.110$ m	Length of the crank
$b = 0.100$ m	Length of the follower
$r = 0.100$ m	Inner radius of the ratchet wheel

Table 2. Inertial properties

Link	**Mass** [kg]	**Inertia** [kgmm^2]
Crank	0.1000	0.0900
Follower	0.0976	0.0853
Ratchet wheel	5.6264	32.7948

$\gamma = 190°$, $\psi = 75°$ and all the links are initially at rest. A driving torque of 1 Nmm acts on the crank. A resisting torque is applied on the ratchet wheel by means of a torsional spring with stiffness 15 Nmm/rad. The impact is assumed anelastic (*i.e.* restitution coefficient e=0) and friction neglected.

The dynamic analysis of the system is executed by means of Gauss-Udwadia–Kalaba formulation and stabilized by Baumgarte criterion with gain parameters α=10, β=1. These values of Baumgarte's coefficients are properly chosen in order to limit the introduction of extraneous frequencies in the time response.

The impact instant t_I is evaluated by monitoring the distance s. When $s = 0$, an event function stops the ODE solver. Then the variation of velocity $\{\Delta\dot{q}\}$ is computed by means of the expression (7). Afterwards the ODE solver restarts the numerical integration with new initial conditions on velocities.

Fig. 5. Ratchet wheel tooth

Fig. 6. Cantilever beam model

Once the dynamic analysis of the system is completed, the stress analysis on the ratchet wheel tooth subjected to impact is performed. For our purposes and with reference to Figure 5, the tooth has been modelled as a cantilever beam with its fixed end corresponding to the inner side of the tooth. The

[1] One hinge is the revolute joint between the frame and the ratchet wheel, the other one is the revolute joint between the crank and the frame.

length h depends on teeth number of the ratchet wheel through the angle θ. The length L of the sloping side, which represents the path of the pawl tip, has been arbitrarily set proportional to h by means of a coefficient $c_{prop} = 1.115$. This will ensure realistic tooth proportions for different teeth number configurations. The length l of the beam is function of two parameters, which have been varied (see section Case studies). These parameters are the number nt of teeth, the angle β_0 between the inner radius of the ratchet wheel and L. In order to simplify the analysis without accuracy loss, the shape of the beam has been always considered as a right–angle triangle. In fact, two different geometries are possible during the analysis according to the angle $\alpha^* = \alpha + \alpha_2$. The configuration for $\alpha^* < 90°$ is the one depicted on the left hand side of Figure 7, otherwise it is the one on the right hand side. The shape of the beam considered neglects the grey part in both sides of Figure 7. This simplification is supported by observing that the grey part has a stiffening effect on the adjacent portion of the beam and thus can be regarded as rigid without introducing a great error. Following the stress recovery methodology discussed in

Fig. 7. Shapes of ratchet wheel tooth

the previous section, a modal analysis has been performed. The first two natural frequencies and the relative modal shapes have been computed. However, the contribution of the second modal shape on the beam dynamic behavior has a little effect. For this purpose the deflection law introduced (see Fig. 6 for nomenclature) is $u(x) = a_1 \frac{x^2}{l^2} + a_2 \frac{x^3}{l^3}$.

Since,

$$A(x) = b(l - x)\tan\alpha \ , \qquad I_n(x) = \frac{b}{12}(l - x)^3 \tan^3\alpha \ . \tag{14}$$

by means of the Rayleigh-Ritz method the first two natural frequencies of the cantilever beam are obtained

$$\Lambda_1 = 1.535 \frac{\sqrt{\rho E}\tan\alpha}{\rho l} \ , \qquad \Lambda_2 = 4.994 \frac{\sqrt{\rho E}\tan\alpha}{\rho l} \ . \tag{15}$$

Now solving system (12) once substituting Λ_1 and then Λ_2, one obtains the values of coefficients a_1 and a_2 for the first and the second beam modal shape, $u_1(x)$ and $u_2(x)$, respectively.

After computing the first two modal shapes, the expression of the dynamic deflection w of the beam is obtained from

$$w = u_1(x)\,\eta_1(t) + u_2(x)\,\eta_2(t) \ . \tag{16}$$

where $\eta_1(t)$ and $\eta_2(t)$ are yet to be determined. For this reason, equations (9) and (10) have been solved once with u_1 and then with u_2, respectively. For both of cases the damping factor of the system has been set equal to $\zeta = 0.03$ which corresponds to the damping factor of steel. Identical initial conditions have been imposed. It is important to impose an initial velocity consistent with the cantilever beam boundary condition. In fact the actual configuration of tooth is such that its inner part, (*i.e.* the one regarded as fixed), has an initial velocity due to the angular velocity of the ratchet wheel at impact time. Modeling the tooth as a cantilever beam may not seem appropriate. However, a possible way to mimic the actual behaviour of the tooth is to consider the *relative* velocity between the free-end and the supposed fixed-end. Thus, the value of v_I has to be the difference between the velocity of the free end and the velocity of the end supposed fixed in the model. Finally, the time varying expression of the bending moment and stress along the cantilever beam is computed by means of equations (13), with $W_b = \frac{1}{6}b\left(l-x\right)^2\tan^2\alpha$ section modulus.

5.1 Case Studies

Two main parameters have been varied during the analysis in order to investigate their influence on the stress distribution along the ratchet tooth. The first one is the number of teeth nt which has been varied in the interval $10 \div 25$ with increments of 5. The second parameter is the angle β_0 (see Figure 5) which varies between $85° \div 105°$ with step 5. In Figure 8 the changing in ratchet wheel shape with the number of teeth is shown ($\beta_0 = 95°$). The other three configurations concerning the angle β_0 display the same behavior reported in Figure 8.

In Figure 9 the relative tooth shapes are presented. Tooth thickness has been set equal to $b = 0.02$ m. For all the case studies, the input parameters of the dynamic analysis are kept constant (*e.g.* input torque, initial configurations, etc.). The output parameters monitored are the impact time t_I, the angular velocity of ratchet wheel after impact $\omega\left(t_I^+\right)$ and the maximum stress σ_{max} registered in the tooth.

Moreover in Figure 10 the comparison between the bending stress along the tooth is reported. In each plot are compared the bending stress curves for the same number of teeth and different values of β_0. The tooth length has been normalized in the range $[0, 1]$. For all the case studies presented, the bending stress attains its maximum at about 40% of the length of the tooth from the fixed end. A chart of the maximum stress as function of the number of teeth and the slope of the tooth is presented in Figure 11. Once known the values of varying parameters, it is possible to predict the maximum bending stress which will occur. Generally, the maximum stress value reduces by increasing the number of teeth but increases with β_0.

Fig. 8. Ratchet wheel shape for different number of teeth ($\beta_0 = 95°$)

Fig. 9. Shape of ratchet wheel tooth for different configurations

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Fig. 10. Stress along the tooth

Fig. 11. Maximum bending stress