A Review of Formulas for the Mechanical Efficiency Analysis of Two Degrees-of-Freedom Epicyclic Gear Trains

Ettore Pennestrì
Pier Paolo Valentini

Università degli Studi di Roma Tor Vergata
Dipartimento di Ingegneria Meccanica
via di Tor Vergata, 110
0133 Roma - Italy

February 18, 2003

Abstract

This paper, after a rigorous proof of the formulas originally proposed by Radzimovsky, demonstrates the numerical equivalence of the different approaches available for computing the mechanical efficiency of two degrees-of-freedom (d.o.f.) epicyclic gear trains. The paper includes also a discussion on the redundancy of data required by some formulas.

Introduction

The 2 d.o.f. epicyclic gear train can be considered the basic unit of many complex gear drives. A mechanical efficiency analysis is a required step in the design process of multi-d.o.f. drives. Technical literature reports many formulas for the mechanical efficiency analysis of basic 2 d.o.f epicyclic gear train (EGT). Deduced from different first principles, but under the same hypotheses, each of these formulas has its own algebraic structure.

After a brief summary of the theoretical steps required to obtain the formulas, their equivalence is demonstrated numerically .

This comparison seems to be novel and helpful for the purpose of shedding further light on an intricate subject.

Also developed herein is a rigorous proof and a new interpretation of Radzimovsky's formulas [1].

The results presented can also be used for the mechanical efficiency analysis of 1 d.o.f. epicyclic gear trains, as described in references [3, 4] or other approaches

[14, 15]. The formulas deduced by Radzimovsky [1], have been embodied by Pennestrì and Freudenstein in their systematic method for mechanical efficiency analysis of EGT [3, 4].

A discussion on computer oriented methods for the analysis of mechanical efficiency in EGT is given by del Castillo [19]. An implementation in PASCAL programming language of the method of Freudenstein and Pennestrì [3, 4] is provided by Ciceroni [18].

All the procedures reviewed:

- consider meshing losses only;
- do not include inertia effects;
- neglect speed and load dependent losses.

The above simplifications have been removed by Mantriota and Pennestrì [16] in a theoretical-experimental investigation on mechanical efficiency analysis of EGTs.

Deduction of mechanical efficiency formulas

Let us consider a basic 2 d.o.f. epicyclic spur-gear train where all the links rotate at constant angular velocity and all losses, except meshing losses, are negligible. Although this hypothesis is very restrictive, it is frequently adopted in industrial pratice (*e.g.* [7, 8]). Moreover, it is also assumed that such losses are not velocity or load dependent. A more general model where load dependent losses are included, has been proposed and experimentally validated in reference [16].

In the functional representation which appears at the left side of Figure 1, there are two external spur gears i and j connected through the gear carrier k. Both j and k are connected through a revolute joint at another link, say the frame m.

To visualize the power-flow through the links, a block representation is often adopted (see right side Figure 1). The unit composed of i, j and k is the *building*

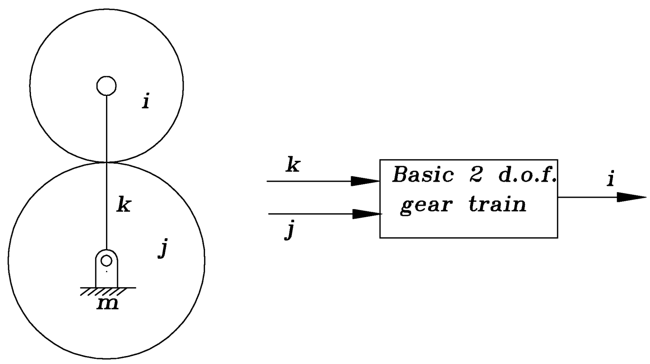

Figure 1: An example of basic 2 d.o.f. EGT with block representation of the power flow.

block of a large class of complex EGTs. Thus the term *basic* appears appropriate to characterize the simplest 2 d.o.f. EGT.

Let us denote with T_j and T_k the torques externally applied on spur gear j and on gear carrier k, respectively. In the absence of friction losses, the application of virtual work principle gives

$$T_i\omega_i' + T_k\omega_k = 0 \ , \tag{1}$$

$$T_i\omega_i'' + T_j\omega_j = 0 \ , \tag{2}$$

where

$$\omega_i' = (1-R)\,\omega_k \ , \tag{3}$$

$$\omega_i'' = R\omega_j \ , \tag{4}$$

are, respectively, the angular velocities of link i when link j or k is held fixed and

$$R = \pm\frac{\text{Number of teeth of gear } j}{\text{Number of teeth of gear } i} \ , \tag{5}$$

is the gear ratio (positive for internally meshing gears).

If both links j and k rotate simultaneously, then the angular velocity of link i is

$$\omega_i = \omega_i' + \omega_i'' \ . \tag{6}$$

This relationship is still valid when meshing losses are taken into account, but some of the previous expressions must be modified. For this purpose, we denote with $\eta_{j(k-i)}$ $\left(\eta_{k(j-i)}\right)$ the mechanical efficiency of the basic EGT when link j (link k) is held fixed.

Because of meshing losses, equations (1) and (2) are respectively modified as follows:

$$T_i\omega_i' + \eta_{j(k-i)}T_k\omega_k = 0 \tag{7}$$

$$T_i\omega_i'' + \eta_{k(j-i)}T_j\omega_j = 0 \ . \tag{8}$$

Thus, adding equations (7) and (8), one obtains the torque at link i

$$T_i = -\left(\eta_{j(k-i)} T_k \frac{\omega_k}{\omega_i} + \eta_{k(j-i)} T_j \frac{\omega_j}{\omega_i}\right) . \tag{9}$$

By definition, the mechanical efficiency η_E of the basic 2 d.o.f. epicyclic gear train is

$$\begin{aligned} \eta_E &= \frac{|T_i \omega_i|}{T_k \omega_k + T_j \omega_j} \\ &= \frac{\eta_{j(k-i)} T_k \omega_k + \eta_{k(j-i)} T_j \omega_j}{T_k \omega_k + T_j \omega_j} . \end{aligned} \tag{10}$$

The reader should observe that the previous expression is the well known formula for computing the mechanical efficiency of a machine composed of two parallel mounted units (see figure 2). In the case under analysis, the mechanical

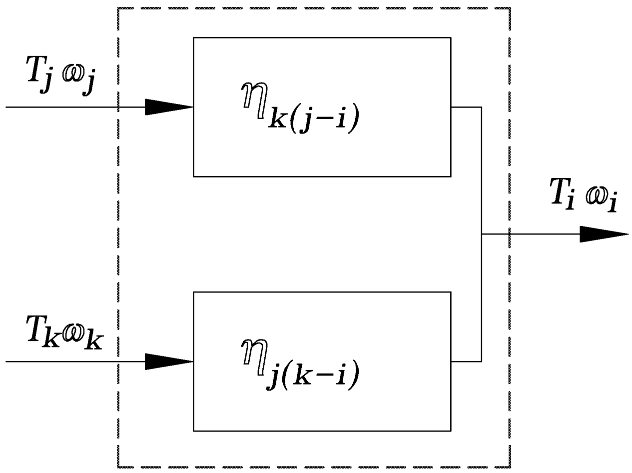

Figure 2: Power flow in a two d.o.f. machine unit

efficiencies of the two units are $\eta_{j(k-i)}$ and $\eta_{k(j-i)}$.

Taking into account (7) and (8), equation (10) can be rewritten as follows:

$$\eta_E = \frac{\omega_i' + \omega_i''}{\dfrac{\omega_i'}{\eta_{j(k-i)}} + \dfrac{\omega_i''}{\eta_{k(j-i)}}} \ . \tag{11}$$

This expression coincides with the one deduced by Radzimovsky [1].

When there is one input link (say wheel i) and two output links (say k and j), the power balance equation is

$$T_i \omega_i \eta_E + T_k \omega_k + T_j \omega_j = 0 \ . \tag{12}$$

Therefore, the mechanical efficiency is

$$\eta_E = \frac{|T_k \omega_k + T_j \omega_j|}{T_i \omega_i} \ . \tag{13}$$

In this case, considering the two kinematic inversions, equations (7) and (8) should be modified, respectively, as follows

$$T_i \omega_i' \eta_{j(k-i)} + T_k \omega_k = 0 \ , \tag{14}$$

$$T_i \omega_i'' \eta_{k(j-i)} + T_j \omega_j = 0 \ . \tag{15}$$

By adding (14) and (15) one obtains

$$T_i \left(\omega_i' \eta_{j(k-i)} + \omega_i'' \eta_{k(j-i)} \right) + T_j \omega_j + T_k \omega_k = 0 \ . \tag{16}$$

By comparing (12) and (16), it follows the expression for mechanical efficiency [1]

$$\eta_E = \frac{\omega_i' \eta_{j(k-i)} + \omega_i'' \eta_{k(j-i)}}{\omega_i} \tag{17}$$

For the purpose of facilitating the use of (11) and (17), the entries of Table 1 have been compiled.

Table 1: Overall efficiency of the basic 2 d.o.f. basic gear train [3, 4].

Input links	Driven links	Overall efficiency η_E
i,k	j	$\dfrac{1}{\dfrac{1}{\left[1+\dfrac{\omega_k(R-1)}{\omega_i}\right]\eta_{k(i-j)}}+\dfrac{1}{\left[1+\dfrac{\omega_i}{\omega_k(R-1)}\right]\eta_{i(k-j)}}}$
j,k	i	$\dfrac{1}{\dfrac{1}{\left[1+\dfrac{\omega_k(1-R)}{\omega_j R}\right]\eta_{k(j-i)}}+\dfrac{1}{\left[1+\dfrac{\omega_j R}{\omega_k(1-R)}\right]\eta_{j(k-i)}}}$
i,j	k	$\dfrac{1}{\dfrac{1}{\left[1-\dfrac{\omega_i}{\omega_j R}\right]\eta_{i(j-k)}}+\dfrac{1}{\left[1-\dfrac{\omega_j R}{\omega_i}\right]\eta_{j(i-k)}}}$
i	k,j	$\dfrac{\eta_{k(i-j)}}{1+\dfrac{\omega_k(1-R)}{\omega_j R}}+\dfrac{\eta_{j(i-k)}}{1+\dfrac{\omega_j R}{\omega_k(1-R)}}$
j	k,i	$\dfrac{\eta_{k(j-i)}}{1+\dfrac{\omega_k(R-1)}{\omega_i}}+\dfrac{\eta_{i(j-k)}}{1+\dfrac{\omega_i}{\omega_k(R-1)}}$
k	i,j	$\dfrac{\eta_{i(k-j)}}{1-\dfrac{\omega_i}{\omega_j R}}+\dfrac{\eta_{j(k-i)}}{1-\dfrac{\omega_j R}{\omega_i}}$

The formulas of efficiency analysis deduced by Radzimovsky have been used in experimental analyses [2].

Application of Radzimovsky's formulas

The formulas of Radzimovsky have been embodied in the systematic procedure of mechanical efficiency analysis proposed by Pennestrì and Freudenstein. In particular, references [3, 4] describe a method based on joint use of these formulas with the analytical findings of Merritt [5] and Macmillan [9]. These authors deduced independently and using different principles, the mechanical efficiencies for all the epicyclic inversions of a basic gear train. In particular, these efficiencies have been related to the gear ratio r and to the mechanical efficiency η of the gear train when operating with a fixed carrier (See Table 2).

Table 2: Mechanical efficiencies of basic gear train epicyclic inversions. $\left| r = \frac{\text{No. of teeth of gear 1}}{\text{No. of teeth of gear 2}} \right| > 1$, $r > 0$: Internal gears, $r < 0$:External gears [5, 9].

Fixed	Input	Driven	Mech.eff.($r > 0$)	Mech.eff.($r < 0$)
k	i	j	η	η
k	j	i	η	η
j	i	k	$\frac{r\eta - 1}{r - 1}$	$\frac{r\eta - 1}{r - 1}$
j	k	i	$\frac{\eta(r - 1)}{r - \eta}$	$\frac{\eta(r - 1)}{r - \eta}$
i	j	k	$\frac{r\eta - 1}{\eta(r - 1)}$	$\frac{r - \eta}{r - 1}$
i	k	j	$\frac{r - 1}{r - \eta}$	$\frac{\eta(r - 1)}{\eta r - 1}$

Alternative approaches

The formulas discussed in this paper have been deduced from different principles of mechanics. Thus, purpose of this section is to give a brief theoretical outline[1] of the reasoning used by other researchers when dealing with the same problem. A more detailed treatment must be found in the original references [5, 6, 8, 10, 17, 11, 12].

Merritt (1947)

Here the method of Merritt [5] (pp.134-135) is described

After subtracting the speed of k, the work put into i and j will be as follows

$$\text{Work put into gear } i = T_i\left(\omega_i - \omega_k\right) \tag{18}$$

$$\text{Work put into gear } j = T_j\left(\omega_j - \omega_k\right) \tag{19}$$

One of these quantities must be positive and the other negative. The positive value refers to the driver link of the basic train under a kinematic inversion which makes the gear carrier fixed. If link i is the driver, then the power loss will be

$$L_i = T_i\left(\omega_i - \omega_k\right)(1-\eta) \ . \tag{20}$$

Similarly, if link j is the driver, then the power loss will be

$$L_j = T_j\left(\omega_j - \omega_k\right)(1-\eta) \ . \tag{21}$$

[1] Whenever possible the original nomenclature will be maintained for a direct comparison.

Since the power loss cannot be affected by a kinematic inversion, the mechanical efficiency of the EGT is

$$\eta_E = \frac{\sum T\omega - T_i\left(\omega_i - \omega_k\right)(1-\eta)}{\sum T\omega} \tag{22}$$

or

$$\eta_E = \frac{\sum T\omega - T_j\left(\omega_j - \omega_k\right)(1-\eta)}{\sum T\omega} \tag{23}$$

where $\sum T\omega$ represents the sum of the positive products of T and ω of the rotating members, *i.e.* the total power input and the appropriate member is selected to determine the loss. The approach of Merritt was used by Hsieh and Tsai in a recent work [14].

Macmillan (1961)

In a basic EGT the following relations always hold

$$T_i + T_j + T_k \quad = 0 \quad , \tag{24}$$

$$\omega_j - \omega_k - R'\left(\omega_i - \omega_k\right) \quad = \quad 0 \ , \tag{25}$$

where

$$R' = R^{-1} \ . \tag{26}$$

Assuming a power flow from gear i to gear j, one can write

$$-P_j = \frac{\eta P_i}{R'} \ . \tag{27}$$

Combining relations (24), (25) and (27) one obtains (see [6], eq. (6))

$$-P_j = \eta P_i + \frac{\eta\left(1 - R'\right)P_k}{\eta - R'} \ . \tag{28}$$

Manipulating the previously mentioned relations one can demonstrate that (see [6], eq.(11))

$$\frac{P_i}{R'} = \frac{P_k \omega_i}{(\eta - R')\,\omega_k} = -\frac{P_j \omega_i}{\eta \omega_3}\ . \tag{29}$$

The total frictional power lost in the EGT is given by

$$\begin{aligned} P_L &= P_i + P_j + P_k \\ &= (1-\eta)\left[P_i - \frac{R'}{\eta - R'} P_k\right] \end{aligned} \tag{30}$$

where all the powers are considered taking into account their algebraic sign. If both P_i and P_k were positive, one would have (see [6], p.40)

$$\eta_E = 1 - \frac{P_L}{P_i + P_k} \tag{31}$$

Therefore, substituting (29) and (30) into (31) one readily obtains[2]

$$\eta_E = 1 - (1-\eta)\left(\frac{1}{P_i' + P_k'}\right)\ , \tag{32}$$

where

$$P_i' = \frac{\omega_i}{\omega_i - \omega_k}\ , \tag{33}$$

$$P_k' = \frac{(\eta - R')\,\omega_k}{R'\,(\omega_i - \omega_k)}\ . \tag{34}$$

Strauch (1970)

For the deduction of the efficiency formulas Strauch[8] starts with the general definition of the mechanical efficiency

$$\eta_E = \frac{P_{ab}}{P_{ab} + V}\ , \tag{35}$$

[2]The reported algebraic substitutions and simplifications are not explicitly mentioned by Macmillan.

where P_{ab} is the output power and V the power loss.

Let ([8], p.39)

$$|P_{iw}| = |T_i(\omega_i - \omega_k)| = |T_j(\omega_j - \omega_k)| \tag{36}$$

$$= |P_{jw}| = |P_{iw}| = |P_w| \tag{37}$$

For an ordinary gear train $P_{ab} = P_w$, thus

$$\frac{V}{P_{ab}} = \frac{1-\eta_{st}}{\eta_{st}} \;, \tag{38}$$

where η_{st} is its efficiency.

For a planetary gear train, we can write

$$\frac{V}{P_{ab}} = \frac{V}{P_w} \cdot \frac{P_w}{P_{ab}} = \frac{P_w}{P_{ab}} \cdot \frac{1-\eta_{st}}{\eta_{st}}. \tag{39}$$

Substituting this into (35) we obtain

$$\eta_E = \frac{1}{1 + \left|\frac{P_w}{P_{ab}} \cdot \frac{1-\eta_{st}}{\eta_{st}}\right|} \;. \tag{40}$$

With reference to the scheme of Figure 3, from (40) Strauch deduces the formulas for the following three different cases ([8], p.45):

- Input link: a. Output links: d and s.

$$(\eta_E)_1 = \frac{1}{1 + \left|k_d \frac{1-J}{J} \cdot \frac{1-\eta_{st}}{\eta_{st}}\right|} \;, \tag{41}$$

- Input links: a and d. Output link: s.

$$(\eta_E)_2 = \frac{1}{1 + \left|k_a \frac{1-J}{1-uJ} \cdot \frac{1-\eta_{st}}{\eta_{st}}\right|} \;, \tag{42}$$

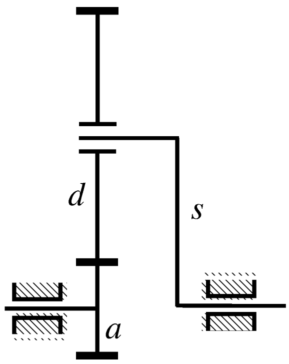

Figure 3: Nomenclature used by Strauch [8].

- Input links: a and s. Output link: d.

$$(\eta_E)_3 = \frac{1}{1 + \left| k_a \left(1 - J\right) \cdot \dfrac{1 - \eta_{st}}{\eta_{st}} \right|} \ , \tag{43}$$

where ([8], pp.38-39)

$$u = \frac{\omega_d - \omega_s}{\omega_a - \omega_s} \ , \tag{44}$$

$$J = \frac{\omega_a}{\omega_d} \ , \tag{45}$$

$$|k_a| = \left| \frac{u}{u - 1} \right| \ , \tag{46}$$

$$|k_d| = \left| \frac{1}{1 - u} \right| \ . \tag{47}$$

Maggiore (1971)

In principle the reasoning followed by Maggiore [10] is similar to the one of Merritt. However, the algebraic structure of the formulas deduced is different.

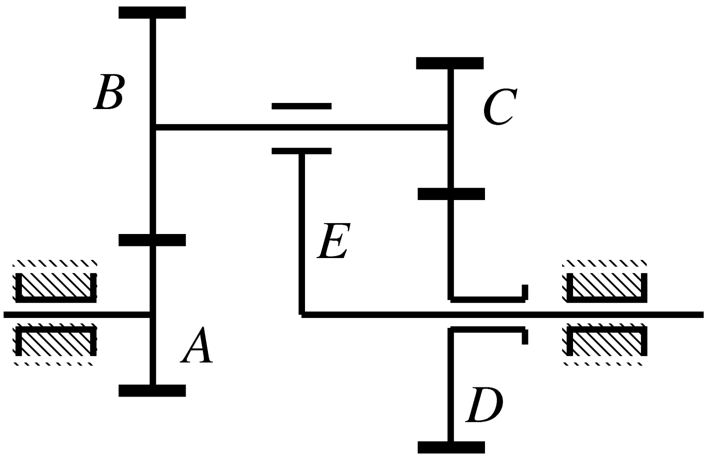

Figure 4: Nomenclature used by Maggiore [10].

With reference to the nomenclature of Figure 4, Maggiore distinguishes the four cases summarized in Table 3. For each case there are two different possibilities (herein denoted as subcases 1 and 2). In particular, subcase 1 applies when, under the kinematic inversion which makes the gear train ordinary, the power flows from gear A to gear D. Subcase 2 applies when, under the same kinematic inversion, the power flows from gear D to gear A.

Under the kinematic inversion which makes the gear train ordinary, the torques ratios are as follows:

- Subcase 1: Driving gear (A), Driven gear (D)

$$\frac{T_A}{T_E} = \frac{\tau_0}{\eta - \tau_0} \ , \quad \frac{T_D}{T_E} = \frac{\eta}{\tau_0 - \eta} \ , \tag{48}$$

- Subcase 2: Driving gear (D), Driven gear (A)

$$\frac{T_A}{T_E} = \frac{\eta\tau_0}{1-\eta\tau_0}\ , \quad \frac{T_D}{T_E} = \frac{1}{\eta\tau_0 - 1}\ ; \tag{49}$$

where T_A, T_D and T_E denote the torques acting on the gears.

Table 3: Combinations of working conditions for the epicyclic gear train shown in Figure 4.

Case No.	Input Gears	Output Gears
I	A-E	D
II	A-D	E
III	A	D-E
IV	E	A-D

According to Maggiore, when we let

$$\tau_0 = \frac{\omega_D - \omega_E}{\omega_A - \omega_E}\ , \tag{50a}$$

$$\tau' = \frac{\omega_E}{\omega_A}\ , \tag{50b}$$

the efficiency of a two d.o.f. epicyclic gear train can be estimated by means of the formulas reported in Table 4, where η denotes the efficiency of the ordinary gear train obtained under a kinematic inversion.

Table 5: Working conditions considered by Monastero [17] in addition to those listed in Table 3

Case No.	Input Gears	Output Gears
V	E-D	A
VI	D	A-E

Table 4: Formulas for computing the efficiency [10].

Case No.	Subcase 1	Subcase 2
I	$\eta_{I,1} = 1 - \dfrac{1-\eta}{1 + \dfrac{\eta\tau'}{\tau_0\left(1-\tau'\right)}}$	$\eta_{I,2} = 1 - \dfrac{1-\eta}{\dfrac{\tau'}{\tau_0\left(\tau'-1\right)} - \eta}$
II	$\eta_{II,1} = 1 - \dfrac{\tau_0\left(1-\eta\right)\left(1-\tau'\right)}{\tau_0 - \eta\left(\tau_0 + \tau' - \tau_0\tau'\right)}$	$\eta_{II,2} = 1 - \dfrac{1-\tau'}{1 + \dfrac{\tau'\left(1-\tau_0\right)}{\tau_0\left(1-\eta\right)}}$
III	$\eta_{III,1} = 1 - \dfrac{1-\eta}{\left(1-\tau'\right)^{-1}}$	$\eta_{III,2} = 1 - \dfrac{1-\eta}{\eta}\left(\tau'-1\right)$
IV	$\eta_{IV,1} = 1 - \dfrac{\tau_0\left(1-\eta\right)\left(\tau'-1\right)}{\tau'\left(\tau_0-\eta\right)}$	$\eta_{IV,2} = 1 - \dfrac{\tau_0\left(1-\eta\right)\left(\tau'-1\right)}{\tau'\left(1-\tau_0\eta\right)}$

Monastero, 1976

With reference to the nomenclature of Figure 4, Monastero [17] considers also, in addition to those listed in Table 3, the working conditions summarized in Table 5.

Thus, to Maggiore's formulas listed in Table 4, Monastero adds those summarized in Table 6. Although all non isomorphic cases have been considered by Maggiore, the formulas added by Monastero may ease the computations.

Table 6: Formulas for computing efficiency presented by Monastero [17] in addition to those listed in Table 4

Case No.	Subcase 1	Subcase 2
V	$\eta_{V,1} = 1 - \frac{(1-\tau')(1-\eta)}{(1-\tau')(1-\eta)-1}$	$\eta_{V,2} = 1 - \frac{(1-\tau')(1-\eta)}{1-\tau'(1-\eta)}$
VI	$\eta_{VI,1} = 1 - \frac{\tau_0(\tau'-1)(1-\eta)}{\eta(\tau'+\tau_0-\tau'\tau_0)}$	$\eta_{VI,2} = 1 - \frac{(1-\eta)\tau_0(1-\tau')}{\tau_0+\tau'-\tau_0\tau'}$

Lucas, 1984

The reasoning of Lucas is similar to that outlined in our demonstration. However, Lucas gives the impression of missing the physical meaning of what he defines as *efficiency factors*. In fact, he states that (see p.169 [12])

> "...the efficiency factor may be greater than unity."

According to what has been demonstrated, the *efficiency factors* coincide with the mechanical efficiencies of epicyclic inversions of a basic gear train (see Table 2). Therefore, these can never be greater than unity.

In particular, reference [12] reports the following expressions

$$\eta_E = \frac{E_{ij} - R_{ij}}{R_{ij}\,(R_{ij}-1)} \left(\frac{\omega_j}{\omega_i} - R_{ij} \right) + \frac{E_{ij}}{R_{ij}} \cdot \frac{\omega_j}{\omega_i} \ , \tag{51}$$

$$\eta_E = \frac{E_{ij}\,(1-R_{ij})\,\omega_j/\omega_i}{R_{ij}\,(1-R_{ij}) + (E_{ij}-R_{ij})\,(\omega_j/\omega_i - R_{ij})} \ , \tag{52}$$

where E_{ij} is the *efficiency factor* and $R_{ij} = R^{-1}$. The expression (52) should be used for the case of input power on both wheel i and gear carrier k. Equation (51) is used for the case of two output power flows and one input power flow.

Henriot, 1979

Henriot [13] starts by stating the torques on the planetary gears for the two different cases:

$$T_j = \begin{cases} -\dfrac{T_i \eta}{R'} & \text{if gear } i \text{ is a driving link,} \\[2ex] -\dfrac{T_i}{R' \eta} & \text{if gear } i \text{ is a driven link.} \end{cases} \tag{53}$$

The torque T_k on the gear carrier is computed by means of the equilibrium equation (12). When the gear carrier k and gear i are both driving links, the mechanical efficiency of the basic EGT is

$$\eta_e = \left| \frac{T_j \omega_j}{T_i \omega_i + T_k \omega_k} \right| . \tag{54}$$

From this general expression, after simplifying a common factor in the numerator and denominator, one readily obtains [3]

$$\eta_e = \frac{\eta \omega_j}{R' \omega_i + (\eta - R')\, \omega_k} . \tag{55}$$

Discussion of a particular case

For the case of the differential shown in Figure 5, some of the formulas deduced could give erroneous results under the following operating conditions[4]:

- driving link: gear carrier k;
- driven links: wheels j and k;

[3] This further step is not explicitly mentioned by Henriot. Moreover, in the original numerical example, the values of the torques were taken into account. As herein demonstrated, within the limit of our hypotheses, since all the torque values cancel out, this is not necessary.

[4] The case discussed in this section has been brought to the attention of the author by Prof. Carlo Innocenti.

- $\omega_i{=}\omega_j$.

For this case, since $R = -1$, the Willis' equation gives

$$\omega_k = \frac{\omega_i + \omega_j}{2} \ . \tag{56}$$

From the last entry of Table 1, when $\omega_i = \omega_j$, one readily obtains

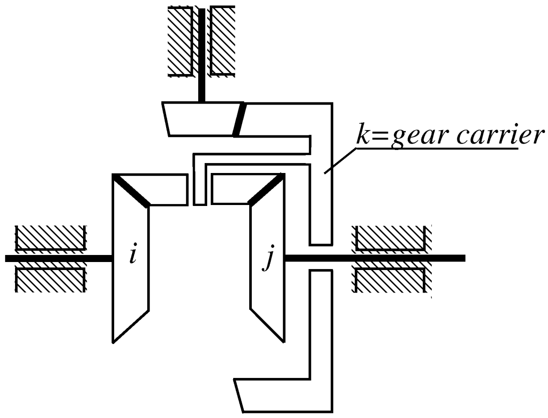

Figure 5: Scheme of a differential gear train: Nomenclature.

$$\eta_E = \frac{\eta_{i(k-j)}}{2} + \frac{\eta_{j(k-i)}}{2} \tag{57}$$

Because of symmetry, it is required that

$$\eta_{i(k-j)} = \eta_{j(k-i)} \ . \tag{58}$$

The equality (58) is mathematically consistent with the third and last entry of Table 2 only for $\eta = 1$. This implies $\eta_{i(k-j)} = \eta_{j(k-i)} = 1$ and, from (57), $\eta_E = 1$. The numerical result just obtained does not contradict the physical reasoning. In fact, under our hypotheses the only source of losses is due to the meshing of gear teeth. When $\omega_i = \omega_j = \omega_k$, there is no meshing and the mechanical efficiency of the differential model must be unity.

Numerical examples

In this section we report the numerical results for the formulas described previously.

All numerical examples refer to the following data[5]:

- Input links: i and k;
- ω_i=10 rad/s , ω_k=-12 rad/s, absolute angular velocities of bodies i and k, respectively;
- $R = -2$, gear ratio;
- $\eta_{st} = \eta$ =0.98, mechanical efficiency of the ordinary gear train whose wheels w_1 and w_2 have, respectively, the absolute angular velocities $\omega_1 = \omega_i - \omega_k$ and $\omega_2 = \omega_j - \omega_k$.

From the kinematic analysis it follows immediately that $\omega_j = -23$ rad/s.

1. **Merritt's formula**

   Let us assume, without of loss of generality, $T_i = 1$, then

$$
\begin{aligned}
T_j &= -T_i \frac{R}{\eta} = 1.96 \\
T_k &= -\left(T_i + T_j\right) = -2.96 \; .
\end{aligned}
$$

[5]The efficiencies are computed using 7 decimal places with the only purpose of demonstrating the equivalence of the formulas considered.

and

$$
\begin{aligned}
P_i &= T_i\omega_i = 10 \\
P_j &= T_j\omega_j = -45.08 \\
P_k &= T_k\omega_k = 35.52 \ .
\end{aligned}
$$

We now compute the expressions (20) and (21), respectively

$$
\begin{aligned}
L_i &= 0.44 \\
L_j &= -0.4312
\end{aligned}
$$

Since L_i, P_i and P_k are all positive, then the efficiency of the differential train is given by formula (22)

$$
\begin{aligned}
\eta_E &= \frac{(P_i + P_k) - L_i}{P_i + P_k} \\
&= \frac{10 + 35.52 - 0.44}{10 + 35.52} = 0.9903339
\end{aligned}
$$

2. **Macmillan's formula**

The equations (26), (33) and (34), with the prescribed numerical data, give

$$
\begin{aligned}
R' &= -\frac{1}{2} \\
P'_i &= \frac{5}{11} \\
P'_k &= \frac{444}{275} \ .
\end{aligned}
$$

By substituting these values into (32) one obtains

$$
\eta_E = 1 - (1 - 0.98)\left(\frac{1}{\frac{5}{11} + \frac{444}{275}}\right) = 0.9903339 \ .
$$

3. **Strauch's formula**

The input links are i and k, thus we need to apply formula (43). By substituting into it the following numerical values

$$\begin{aligned} J &= -\frac{10}{23} \\ u &= -\frac{1}{2} \\ k_a &= \frac{1}{3} , \end{aligned}$$

one obtains

$$(\eta_E)_3 = 0.9903339 .$$

4. **Maggiore's formula**

The adaptation of our scheme to the one used by Maggiore (see Figure 4) requires that $\omega_A = \omega_D$, thus, from (50), we obtain

$$\tau_0 = 1 ,$$

$$\tau' = \frac{12}{23} .$$

Considered the kinematic conditions of the epicyclic gear train under analysis, the efficiency formula (case I-subcase 1), from Table 4 applies. By substituting the ratios just computed one readily obtains

$$\eta_{I,1} = 0.9903339$$

5. **Radzimovsky's formula**

Considered the value of the gear ratio R, from Table 2, follows:

$$\eta_{i(k-j)} = \frac{\eta\,(R-1)}{\eta R-1} = 0.9932432\ , \tag{59}$$

$$\eta_{k(i-j)} = \eta = 0.98\ . \tag{60}$$

Since

$$1+\frac{\omega_k\,(R-1)}{\omega_i} = \frac{23}{5}\ ,$$
$$1+\frac{\omega_i}{\omega_k\,(R-1)} = \frac{23}{18}\ ,$$

by applying the first formula of Table 1, one obtains:

$$\eta_E = \frac{1}{\frac{1}{\frac{23}{5}0.98}+\frac{1}{\frac{23}{18}0.993243}} = 0.9903339\ . \tag{61}$$

6. **Lucas' formula**

In the case under analysis $E_{ij} = \eta = 0.98$ and R_{ij}=-0.5, therefore the formula (52) gives:

$$\eta = \frac{0.98\,(1+0.5)\cdot(-23)\cdot 0.1}{-0.5\,(1+0.5)+(0.98+0.5)\,[(-23)\cdot 0.1+0.5]} = 0.9903339$$

7. **Henriot's formula**

Since ω_j =-23, $R' = -0.5$, equation (55) gives

$$\eta_e = \frac{0.98\cdot(-23)}{-0.5\cdot 10-(0.98+0.5)} = 0.9903339$$

1 Conclusions

With reference to the problem of computing the mechanical efficiency of a 2 d.o.f. basic EGT, a new analysis of the formulae originally deduced by Radzimovsky has been given. This analysis, based on the principle of virtual work, allows interpretation of the Radzimovsky's formula as a particular case of the more general expression for computing the efficiency of two parallel mounted units. Also, derivation of six other methods described and the results for an example problem using all methods was provided. Notwithstanding the apparent algebraic differences in the formulas presented, it has been shown that the different approaches are numerically equivalent.

References

[1] Radzimovsky, E.I., 1959, How to Find Effiicency, Speed and Power in Planetary Gear Drives, *Machine Design*, June 11, pp.144-153.

[2] Kausba, R., Radzimovsky, E.I., 1973, A Multi-Purpose Planetary Gear Testing Machine for Studies of Gear Drive Dynamics, Efficiency, and Lubrication, *ASME Journal of Engineering for Industry*, pp.1123-1130.

[3] Pennestrì, E., 1991, On the Automatic Design Analysis of Gear Trains, Doctoral Dissertation, Columbia University, Available from U.M.I., Order n.9127952.

[4] Pennestrì, E., Freudenstein, F., 1993, The Mechanical Efficiency of Epicyclic Gear Trains, ASME Journal of Mechanical Design, vol.115, pp.645-651.

[5] Merritt, H.E., 1947, *Gears*, Sir I. Pitman & Sons, London.

[6] Macmillan, R.H., 1961, Power Flow and Loss in Differential Mechanisms, Journal of Mechanical Engineering Science, vol.3, n.1, pp. 37-41.

[7] Looman, , J., 1970, *Zahnradgetriebe*, Springer Verlag, New York.

[8] Strauch, H., 1970, *Theorie und Praxis der Planetengetriebe*, Krauskopf-Verlag.

[9] Macmillan, R.H., 1949, Epicyclic Gear Efficiencies, The Engineer, Dec.23rd, pp.727-728.

[10] Maggiore, A., 1971, The efficiency of epicyclic two d.o.f. gear trains (in italian), *Atti I Congresso Nazionale di Meccanica Teorica ed Applicata*, Udine, vol.III, pp.65-85.

[11] Lucas, G.G., 1984, Expressions governing the power flow through differential mechanisms having imperfect efficiencies, Proc. Institution of Mechanical Engineers, C12/84 Driveline 84.

[12] Lucas, G.G., 1986, *Road Vehicle Performance*, Gordon and Breach Science Publishers, Transportation Studies vol.7, pp.166-172.

[13] Henriot, G., 1979, *Traité théorique et pratique des engranages*, 6^{th} ed., Bordas, Paris (see p. 609-612 of the italian translation published by Tecniche Nuove).

[14] Hsieh, H.L., Tsai, L.W., 1998, The selection of a Most Efficient Clutching Sequence Associated with Automatic Transmissions, *ASME Journal of Mechanical Design*, vol.120, pp.514-519.

[15] Hedman, A., 1988, *Mechanical Transmission Systems- a General Computer-based method of Analysis*, Chalmers University of Technology, Division of Machine Elements, Göteborg.

[16] Mantriota, G., Pennestrì, E., 2001, Theoretical and Experimental Efficiency Analysis of Multi Degrees-of-Freedom Epicyclic Gear Trains, *Multibody System Dynamics*, vol.7, 2002, pp.389-406.

[17] Monastero, R., 1976, The efficiency of series connected epicyclic gear trains *in italian*, Rivista Associazione Tecnica dell'Automobile (ATA), Torino, Italy, Luglio-Agosto.

[18] Ciceroni, N., 1989, *A systematic procedure for mechanical efficiency analysis of epicyclic gear trains*, (in italian), Tesi di Laurea, Università degli Studi di Roma Tor Vergata, Academic Year 1998-1999.

[19] del Castillo, J.M., 2002, The analytical expression of the efficiency of planetary gear trains, *Mechanism and Machine Theory*, vol.37, pp.197-214.

Acknowledgement

The author gratefully acknowledges the financial support from Ministero dell'Università e della Ricerca Scientifica.