A Review of Multibody Dynamics Formulations

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Review of *classic* multibody dynamics formulations.

Compare the results of numerical tests on the reviewed and two new dynamic formulations.
Presentation outline

- Review of *classic* multibody dynamics formulations.
- Compare the results of numerical tests on the reviewed and two new dynamic formulations.
The set of coordinates in multibody dynamics

1. Minimal set of coordinates (ODE)
2. Redundant set of coordinates (DAE)
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1. Minimal set of coordinates (ODE)
2. Redundant set of coordinates (DAE)
The multibody equations of dynamics

Differential-Algebraic Equations (DAE) system (differential index 3)

\[
[M] \{\ddot{q}\} + \left[\psi_q^T\right] \{\lambda\} = \{F_e\} \quad (1a)
\]
\[
\{\psi\} = 0 \quad (1b)
\]
The multibody equations of dynamics

Differential-Algebraic Equations (DAE) system (differential index 1)

\[
[M] \{\ddot{q}\} + \left[\psi_q^T\right] \{\lambda\} = \{F_e\} \tag{2a}
\]
\[
[\psi_q] \{\dddot{q}\} = \{\gamma\} \tag{2b}
\]
Common problems

1. Positions and velocity constraints are not necessarily satisfied by the computed numerical solution (solution drift).

2. The constraints equations may not be independent. (i.e. Jacobian without a full rank).

3. The mass matrix could be singular.
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Numerical integration procedures of multibody dynamics DAE systems

1. Coordinate partitioning method.
4. Dynamic formulations deduced from the least squares criterion.
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1. Coordinate partitioning method.
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The coordinate partitioning method

The set of coordinates is partitioned as follows:

\[ \{q\}_{n \times 1} = \begin{bmatrix} v \\ u \end{bmatrix} \]  \hspace{1cm} (3)

where
- \[ \{u\}_{(n-F) \times 1} = \text{dependent coordinates} \]
- \[ \{v\}_{F \times 1} = \text{independent coordinates} \]
The coordinate partitioning method

Equation of dynamics in partitioned form (Haug, Wehage, 1982)

\[
\begin{align*}
[\mathbf{M}^{vv}] \{\ddot{v}\} + [\mathbf{M}^{vu}] \{\ddot{u}\} + [\Psi_v]^T \{\lambda\} &= \{Q^v\} \quad (4) \\
[\mathbf{M}^{uv}] \{\ddot{v}\} + [\mathbf{M}^{uu}] \{\ddot{u}\} + [\Psi_u]^T \{\lambda\} &= \{Q^u\} \quad (5) \\
[\Psi_u]_{m \times (n-F)} \{\ddot{u}\} + [\Psi_v]_{m \times F} \{\ddot{v}\} &= \{\gamma\} \quad (6)
\end{align*}
\]
The coordinate partitioning method

Eliminating $\lambda$ and $\ddot{u}$ one obtains

$$\begin{bmatrix} \hat{M} \end{bmatrix}_{F \times F} \{ \ddot{v} \}_{F \times 1} = \{ \hat{Q} \}_{F \times 1}$$  \hspace{1cm} (7)

where

$$\begin{bmatrix} \hat{M} \end{bmatrix} = [M^{vv}] - [M^{vu}] [\psi_u]^{-1} [\psi_v]$$

\hspace{1cm} - $[\psi_v]^T ( [\psi_u]^{-1} )^T ( [M^{uv}] - [M^{uu}] [\psi_u]^{-1} [\psi_v] )$ \hspace{1cm} (8)

and

$$\begin{bmatrix} \hat{Q} \end{bmatrix} = \{ Q^v \} - [M^{vu}] [\psi_u]^{-1} \{ \gamma \}$$

\hspace{1cm} - $[\psi_v]^T ( [\psi_u]^{-1} )^T ( \{ Q^u \} - [M^{uu}] [\psi_u]^{-1} \{ \gamma \} )$ \hspace{1cm} (9)
Maggi like methods - Orthogonalization of constraints

Figure: Gian Antonio Maggi, Milano (1856-1937)
[\mathbf{M}] \{\ddot{q}\} + [\psi_q]^T \{\lambda\} = \{Q\} \tag{10}

The idea behind the orthogonalization of constraints is the elimination of reaction forces.

For this purpose we need a matrix \([V]\) such that:

\[
\{\dot{q}\} = [V] \{\dot{v}\}
\]

\[
[\psi_q] [V] = [0]
\]
\[
[M] \{\ddot{q}\} + [\psi_q]^T \{\lambda\} = \{Q\} \tag{10}
\]

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\[
\{\dot{q}\} = [V] \{\dot{v}\}
\]

\[
[\psi_q] [V] = [0]
\]
Since

\[
\{\ddot{q}\} = [V] \{\ddot{v}\} + [\dot{V}] \{\dot{v}\}
\]

the equation of dynamics \([M] \{\ddot{q}\} + [\Psi_q]^T \{\lambda\} = \{Q\}\) are transformed into

\[
\]  
(11)
How to compute $\dot{V} \{ \dot{V} \}$

Transform $[\Psi_q] \{ \ddot{q} \} = \{ \gamma \}$ into

$$
[\Psi_u] \{ \ddot{u} \} + [\Psi_v] \{ \ddot{v} \} = \{ \gamma_u \} \tag{12}
$$

$$
\{ \ddot{q} \} = \left\{ \begin{array}{c}
\ddot{u} \\
\ddot{v}
\end{array} \right\} = \left[ -[\Psi_u]^{-1} [\Psi_v] \right] \{ \ddot{v} \} + \left[ [\Psi_u]^{-1} \{ \gamma_u \} \right] \{ 0 \} \tag{13}
$$
Transform $[\Psi_q] \{\ddot{q}\} = \{\gamma\}$ into

$$[\Psi_u] \{\ddot{u}\} + [\Psi_v] \{\ddot{v}\} = \{\gamma_u\} \tag{12}$$

$$\{\ddot{q}\} = \begin{Bmatrix} \ddot{u} \\ \ddot{v} \end{Bmatrix} = \begin{bmatrix} -[\Psi_u]^{-1} &[\Psi_v] \\ [I] & [0] \end{bmatrix} \{\ddot{v}\} + \begin{Bmatrix} [\Psi_u]^{-1} \{\gamma_u\} \\ \{0\} \end{Bmatrix} \tag{13}$$
1. The set \{v\} of independent coordinates and the matrix \[V\] is not unique.
2. There are different numerical methods for the computation of matrix \[V\]
Numerical methods for computing the matrix $[V]$

2. QR decomposition (Kim and Vanderploeg, 1986)
3. Singular value decomposition (SVD), (Mani and Haug, 1985), Likins and Singh (1985)
4. The PUTD decomposition (Amirouche et al., 1988)
5. The QTZ decomposition (Vita and Pennestrì, 2004)
6. The Schur decomposition (Pennestrì and Valentini, 2007)
Let us form the matrix

\[ [H] = [\psi_q]^T [\psi_q] \]  \hspace{1cm} (14)

Since \([H]_{n \times n}\) is symmetric, all the eigenvalues are positive and there exists the similarity transform

\[ [T]^T [H] [T] = [\Lambda] \]  \hspace{1cm} (15)

where \([T]\) is the orthogonal matrix of eigenvectors and \([\Lambda]\) the diagonal matrix of eigenvalues.
The matrix \([T]\) can be partitioned as follows

\[
[T] = \begin{bmatrix}
[T_1]_{n \times (n-r)} & [T_2]_{n \times r}
\end{bmatrix}
\]  

where \([T_1]\) is the matrix of the eigenvectors associated to zero eigenvalues.

It can be demonstrated the condition of orthogonality

\[
[\Psi_q] [T_1] = [0]
\]

\[
\therefore [V] = [T_1]
\]
The matrix \([ T ]\) can be partitioned as follows

\[
[T] = \begin{bmatrix}
[T_1]_{n \times (n-r)} & [T_2]_{n \times r}
\end{bmatrix}
\]  \(16\)

where \([ T_1 ]\) is the matrix of the eigenvectors associated to zero eigenvalues.

It can be demonstrated the condition of orthogonality

\[
[\Psi_q] [T_1] = [0]
\]  \(17\)

\[
\therefore [V] = [T_1]
\]
\[
[\Psi_q]^T = \begin{bmatrix} [Q_1]_{n \times (n-F)} & [Q_2]_{n \times F} \end{bmatrix} \begin{bmatrix} [R_1]_{r \times m} \\ [0]_{F \times m} \end{bmatrix} = [Q_1] [R_1]
\]

where \([Q_1]\) and \([Q_2]\) simultaneously satisfy the conditions

\[
[Q_2]^T [Q_1] = [0] \quad \text{(19)}
\]

\[
[Q_2]^T [Q_2] = [I] \quad \text{(20)}
\]
\[ [Q_2]^T [\psi_q]^T = [Q_2]^T [Q_1] [R_1] = [0] \] (21)

\[ \therefore [V] = [Q_2] \]
SVD method - Singh and Likins (1985), Mani and Haug (1985)

\[
[\Psi_q]^T = [W][\Lambda][U] \quad (22)
\]

\[
\begin{bmatrix}
[W_d]_{n \times r} & [W_i]_{n \times (n-r)}
\end{bmatrix}
\begin{bmatrix}
[\Lambda_1]_{r \times r} & [0]_{(n-r) \times r}
\end{bmatrix}
[U]^T_{r \times r}
\]

\[
= [W_d][\Lambda_1][U]^T \quad (23)
\]

where \([W_i]^T[W_d] = [0]\)
SVD method (cont.)

\[
\]

\[
\therefore [V] = [W_i]
\]
The method’s first step is to obtain the Householder transform matrix $[H]$ of $[\psi_q]^T$. Then, through the Gram-Schmidt process, it is identified a matrix $[D_2]$ such that

$$[D_2]_{(n-r) \times n} [H]_{n \times n} [\psi_q]^T = [0]$$

(25)

$$\therefore [V] = [H] [D_2]^T$$
The matrix $[D_2]$ has the following block structure

$$[D_2] = \begin{bmatrix} [0]_{(n-r) \times r} & [I]_{(n-r) \times (n-r)} \end{bmatrix}$$

(26)

One can verify that $[D_2][H]^T = [Q_2]^T$. 
The Schur decomposition of the matrix \([H]\) has the form

\[
[K]^T [H] [K] = [U]
\]  \hspace{1cm} (27)

where \([K]\) is a unitary matrix and \([U]\) an upper triangular matrix. The eigenvalues of \([H]\) are on the diagonal of \([U]\).

The matrix \([K]\) can be partitioned as follows

\[
[K] = \begin{bmatrix} [K_1]_{n \times (n-r)} & [K_2]_{n \times r} \end{bmatrix}
\]  \hspace{1cm} (28)

where \([K_1]\) are the generalized Schur vectors associated to zero eigenvalues.
The Schur decomposition of the matrix $[H]$ has the form

$$[K]^T [H] [K] = [U]$$  \hspace{1cm} (27)$$

where $[K]$ is a unitary matrix and $[U]$ an upper triangular matrix. The eigenvalues of $[H]$ are on the diagonal of $[U]$. The matrix $[K]$ can be partitioned as follows

$$[K] = egin{bmatrix} [K_1]_{n \times (n-r)} & [K_2]_{n \times r} \end{bmatrix}$$  \hspace{1cm} (28)$$

where $[K_1]$ are the generalized Schur vectors associated to zero eigenvalues.
It can be easily verified the condition of orthogonality

\[
[\Psi_q] [K_1] = [0] \tag{29}
\]

\[
\therefore [V] = [K_1]
\]
From Gauss’ Least Action Principle one obtains

\[ \{ \ddot{q} \} = \{ \ddot{q}_f \} + [M]^{-\frac{1}{2}} [D]^+ (\{ \gamma \} - [\Psi_q] \{ \dot{q}_f \}) \] . \quad (30)

where

\[ \{ \ddot{q}_f \} = [M]^{-1} \{ F \} \] , \quad (31)

and

\[ [D] = [\Psi_q] [M]^{-\frac{1}{2}} \] , \quad (32)

with

\[ [M]^{-1} = [M]^{-\frac{1}{2}} [M]^{-\frac{1}{2}} \] . \quad (33)
Udwadia-Phohomsiri equation (2006)

\[ \{ \ddot{q} \} = \left[ \bar{M} \right]^+ \left\{ \begin{array}{c} F \\ \gamma \end{array} \right\} \]  \hspace{1cm} (34)

where

\[ \begin{bmatrix} \bar{M} \end{bmatrix} = \begin{bmatrix} \left( I - \psi_q^+ \psi_q \right) M \\ \psi_q \end{bmatrix} \]  \hspace{1cm} (35)

Note: It can be used also when the mass matrix is singular.
\[
\begin{bmatrix}
M & \Psi_q^T \\
\Psi_q & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{q} \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
Q \\
\gamma
\end{bmatrix}
\] (36)
The pseudoinverse of the matrix of coefficients in the equation of dynamics can be expressed in the form

$$\begin{bmatrix} M & \psi_q^T \\ \psi_q & 0 \end{bmatrix}^+ = \begin{bmatrix} M^{-1} - M^{-1}\psi_q^T Q^+ \psi_q M^{-1} & M^{-1}\psi_q^T Q^+ \\ Q^+ \psi_q M^{-1} & -Q^+ \end{bmatrix}.$$  \hspace{1cm} (37)

where

$$[Q] = [\psi_q] [M]^{-1} [\psi_q]^T.$$  \hspace{1cm} (38)
\[ \{\ddot{q}\} = ([I] - [H][\psi_q]) [M]^{-1} \{F\} + [H] \{\gamma\} \]  

where

\[ [H] = [M]^{-1} [\psi_q]^T [Q]^+ , \]  

\[ [Q] = [\psi_q] [M]^{-1} [\psi_q]^T . \]
\[
\begin{bmatrix}
M & \psi_q^T \\
\psi_q & 0
\end{bmatrix}^+ = \begin{bmatrix}
0 & \psi_q^+ \\
(\psi_q^+)^T & - (\psi_q^+)^T [R]
\end{bmatrix}
+ \begin{bmatrix}
I \\
-R^T
\end{bmatrix} [Q] \begin{bmatrix}
I \\
-R
\end{bmatrix}.
\]

where

\[
[E] = [I] - [\psi_q]^+ [\psi_q],
\]

\[
[Q] = ([E] [M] [E])^+, \]

\[
[R] = [M] [\psi_q]^+,
\]

(42) (43) (44)
\[ \{\ddot{q}\} = [Q] \{F\} + ([\psi_q]^+ - [Q][M][\psi_q]^+) \{\gamma\} . \quad (45) \]

Note: It can be used also when the mass matrix is singular.
The mechanical system analyzed
Problem dimension

- Generalized coordinates - \{q\}: 24
- Mass matrix - \([M]\): 24 × 24
- Jacobian matrix - \([\Psi_q]\): 24 × 24 (rank: 22)
- Force vector - \{F\}: 24
Angular displacement

Time (s)

Theta

Angular displacement

0 1 2 3 4 5 6 7 8 9 10

3.8

4

4.2

4.4

4.6

4.8

5

5.2

5.4

5.6

5.8

Time (s)

Theta

Angular displacement

de Falco, Pennestrì, Guida

Multibody dynamics review
All computations using standard MATLAB libraries.
In the subroutines only array operations have been used.
Numerical integration by means of \texttt{ODE45} with
\texttt{RelTol}=10^{-6}, \texttt{AbsTol}=10^{-6}
Rank and Gaussian elimination using 10^{-6} as a threshold
value for zero.
Numerical results

Table: Comparison of computational efficiency.

<table>
<thead>
<tr>
<th>Method</th>
<th>CPU Time (s)</th>
<th>Function calls</th>
<th>ODE Solver (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero Eigenvalue</td>
<td>66.2</td>
<td>1840</td>
<td>0.38</td>
</tr>
<tr>
<td>Coordinate partitioning</td>
<td>128</td>
<td>1846</td>
<td>0.47</td>
</tr>
<tr>
<td>QR</td>
<td>66.1</td>
<td>1840</td>
<td>0.38</td>
</tr>
<tr>
<td>SVD</td>
<td>Fails</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PUTD</td>
<td>66.5</td>
<td>1840</td>
<td>0.38</td>
</tr>
<tr>
<td>Schur</td>
<td>66.7</td>
<td>1828</td>
<td>0.38</td>
</tr>
<tr>
<td>Udwadia-Kalaba</td>
<td>2.65</td>
<td>1954</td>
<td>0.36</td>
</tr>
<tr>
<td>Udwadia-Phomsiri</td>
<td>3.88</td>
<td>1828</td>
<td>0.38</td>
</tr>
<tr>
<td>de Falco et al. (1)</td>
<td>2.61</td>
<td>1828</td>
<td>0.36</td>
</tr>
<tr>
<td>de Falco et al. (2)</td>
<td>3.60</td>
<td>1828</td>
<td>0.43</td>
</tr>
</tbody>
</table>
A review and comparison of different multibody dynamics formulations has been presented.

The numerical accuracy seems not influenced by the method used for the orthogonalization. However, this may not be necessarily true for large scale problems.

Rank computation problems.

Iterative algorithms for the algebra decompositions should be used.
Conclusions

1. A review and comparison of different multibody dynamics formulations has been presented.
2. The numerical accuracy seems not influenced by the method used for the orthogonalization. However, this may not be necessarily true for large scale problems.
3. Rank computation problems.
4. Iterative algorithms for the algebra decompositions should be used.
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2. The numerical accuracy seems not influenced by the method used for the orthogonalization. However, this may not be necessarily true for large scale problems.

3. Rank computation problems.

4. Iterative algorithms for the algebra decompositions should be used.
MATLAB CODE
function [ddq] = zeroeigenvalueform(M, Psiq, gamma, f)

% It is assumed that the dynamics equations are in the following form
% | M   Psiq^T |   ddq |   f |
% |       |       | =
% | Psiq  0   | lambda | gamma |

[m, n] = size(Psiq);
[BBg, ipoint1] = rref([Psiq gamma], 1.e-3);
r = length(ipoint1);
Psiqm = BBg(1:r, 1:n);
[T, S] = eig(Psiqm' * Psiqm);
V = T(1:n, 1:(n-r));
BB = V';

gammau = BBg(1:n, n+1);
Sgamma = ([Psiqm; BB]) \ gammau;

Mh = V' * M * V;
Fh = V' * f - V' * M * Sgamma;

ddv = Mh \ Fh;

ddq = V * ddv + Sgamma;
end
end of function zeroeigenvalueform
function [ddq]=svdform(M, Psiq, gamma, f)
% Purpose: Computation of accelerations in a multibody dynamics simulation
% by means of svd formulation (Mani-Haug 1986).
% It is assumed that the dynamics equations are in the following form:
% \[
% \begin{pmatrix}
% M & \text{Psiq}^T \\
% \text{Psiq} & 0
% \end{pmatrix}
% \begin{pmatrix}
% \text{ddq} \\
% \lambda
% \end{pmatrix}
% =
% \begin{pmatrix}
% f \\
% \text{gamma}
% \end{pmatrix}
% \]
[m,n]=size(Psiq);
[BBg,ipoint1]=rref([Psiq gamma],1.e-3);
r=length(ipoint1);
[Uh,S1,Vh] = svd(Psiq');
V=Uh(1:n,r+1:(n));
% Compute \([S]*\{\text{gamma}\}\)
BB=V'; [mbb,nbb]=size(BB); Psiqm=BBg(1:r,1:n);
gammau=BBg(1:n,n+1);
Sgamma=([Psiqm;BB])\gammau;
Mh=V'*M*V; Fh=V'*f-V'*M*Sgamma;
ddv=Mh\Fh; ddq=V*ddv+Sgamma;
end %end of function svdform
function [ddq] = qrform(M, Psiq, gamma, f)
% Purpose: Computation of accelerations in a multibody dynamics
% by means of qr formulation (Kim-Vanderploeg 1986).
% It is assumed that the dynamics equations are in the following form
% _ _
% | M Psiq^T | | ddq | | f |
% | | =
% | Psiq 0 | | lambda | | gamma |
[m,n] = size(Psiq);
[BBg, ipoint1] = rref([Psiq gamma], 1.e-3);
r = length(ipoint1);
Psiqm = BBg(1:r, 1:n);
[QQ, R] = qr(Psiqm');
V = QQ(1:n, r+1:n);
BB = V'; gammau = BBg(1:n, n+1);
Sgamma = ([Psiqm BB]) \ gammau;
Mh = V' * M * V;
Fh = V' * f - V' * M * Sgamma;
ddv = Mh \ Fh;
ddq = V * ddv + Sgamma;
end % end of function qrform
function [ddq]=putdform(M,Psiq,gamma,f)
% Purpose: Computation of accelerations in a multibody dynamics by means of putd formulation
% It is assume that the dynamics equations are in the following form
%                     _ _          _ _          _ _          _ _
%          _______    _______    _______    _______    _______    _______  
% | M  Psiq^T |   | ddq |   | f |          |            |
% |        |   |    |   |  =          |            |
% | Psiq   0 |   | lambda |   | gamma |

[m,n]=size(Psiq);
[BBg,ipoint1]=rref([Psiq gamma],1.e-3);
r=length(ipoint1);
Psiqm=BBg(1:r,1:n);
[H,R]=qr(Psiqm');
D22=eye(n-r,n-r);D21=zeros(n-r,r);
D2T=[D21 D22]; V=H*D2T';
BB=V'; gammau=BBg(1:n,n+1);
Sgamma=([Psiqm BB])\gammau;
Mh=V'*M*V; Fh=V'*f-V'*M*Sgamma;

ddv=Mh\Fh; ddq=V*ddv+Sgamma;
end %end of function putdform
function [ddq]=udwadiakalaba (M, Psiq, gamma, f)
% Purpose: Computation of accelerations in a multibody dynamics
% by means of the Udwadia-Kalaba formulation.
% It is assumed that the dynamics equations are in the following form
% _ _
% | M  Psiq^T | | ddq | | f |
% | | | | =
% | Psiq 0 | | lambda | | gamma |
% _ _

v=diag (M);
v=v.^-1;
v1=v.^1/2;
M inv=diag (v);
M inv12=diag (v1);
ddq f=M inv*f;
D=Psiq*M inv12;
D ps=p inv (D, 1.e-6);
a1=(gamma-Psiq*ddq f);
ddq=ddq f+M inv12*D ps*a1;
end % end of function udwadiakalaba
function [ddq] = schurform(M, Psiq, gamma, f)
% Purpose: Computation of accelerations in a multibody dynamics simulation
% by means of qr formulation (Pennestri'-Valentini, 2007).
% It is assume that the dynamics equations are in the following form:
% |
% | M  Psiq^T  | | ddq | | f |
% | | | =
% | Psiq  0 | | lambda | | gamma |
[m, n] = size(Psiq);
[BBg, ipoint1] = rref([Psiq gamma], 1.e-3);
r = length(ipoint1);
[US, K] = schur(Psiq' * Psiq);
V = US(1:n, 1:(n-r)); BB = V';
[mbb, nbb] = size(BB);
Psiqm = BBg(1:r, 1:n);
gammau = BBg(1:n, n+1);
Sgamma = ([Psiqm; BB]) \ gammau;
Mh = V' * M * V;
Fh = V' * f - V' * M * Sgamma;
ddv = Mh \ Fh;
ddq = V * ddv + Sgamma;
end %end of function schurform
function [ddq] = udwadia2 (M, Psiq, gamma, f)
% Purpose: Computation of accelerations in a multibody dynamics simulation
% by means of Udwadia-Phohomsiri.
% It is assume that the dynamics equations are in the following form:
%
% | M   Psiq^T | | ddq | | f |
% |               | |     | =
% | Psiq  0      | | lambda | | gamma |
%
[m, n] = size(Psiq);
Psiqp = pinv(Psiq, 1.e-6);
E = (eye(m) - Psiqp * Psiq) * M;
ddq = pinv([E; Psiq], 1.e-6) *[f; gamma];
end % of function udwadia2
function [ddq] = mio(M, Psiq, gamma, f)
% Purpose: Computation of accelerations in a multibody dynamics simulation by means of a new dynamic formulation.
% It is assumed that the dynamics equations are in the following form:
% _ _
% | M  Psiq^T | | ddq | | f |
% |       | |    | =
% | Psiq   0  | | lambda | | gamma |
% _ _

v = diag(M);
v = v.^-1;
M inv = diag(v);
Q = Psiq * M inv * Psiq';
MQ = pinv(Q, 1.e-6);
ddq f = M inv * f;
H = M inv * Psiq' * MQ;
ddq = (ddq f - H * Psiq * ddq f) + H * gamma;
end % of function mio
function [ddq] = mioultra (M, Psiq, gamma, f)
% Purpose: Computation of accelerations in a multibody dynamics simulation by means of a new dynamic formulation.
% It is assumed that the dynamics equations are in the following form:
% _ _
% | M  Psiq^T | | ddq | | f |
% | | | | =
% | Psiq 0 | | lambda | | gamma |
% _ _
[m, n] = size (M);
Psiqp = pinv (Ppsiq, 1.e-6);
R = M * Psiqp;
Id = eye (m);
E = (Id - Psiqp * Psiq);
Q = pinv (E * M * E, 1.e-6);
ddq = Q * f + (Psiqp - Q * M * Psiqp) * gamma;
end % of function mioultra