Dynamic Analysis of Epicyclic Gear Train by Means of Computer Algebra

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Abstract

The paper illustrates the application of a systematic technique for the deduction of dynamic equations of epicyclic gear trains. The use of the proposed method is particularly amenable in conjuction with procedures for the computerized enumeration of geared kinematic chains. The discussed examples illustrate how the presented approach can be extended for computing the equivalent inertia of an epicyclic gear train or for the stability analysis of 2 d.o.f. gear trains.

*Keywords:*Epicyclic gear trains, gear trains multibody dynamics, stability analysis.

1 Introduction

Structural synthesis of gear drives by means of graph theory has been an important area of investigation in the field of kinematics. Although this theory is a tool frequently used in multibody dynamics, and distinguished researchers contributed to its development (e.g. [1, 2, 3]), pioneer analyses on the kinematic structure of mechanisms using graph theory are due to Freudenstein and his coworkers [4, 5].

The first procedure for the enumeration of geared kinematic chains (GKC), partly computerized and partly based on inspection, is due to Buchsbaum and Freudenstein [6]. Their landmark paper disclosed also noteworthy properties. The most important, which has been extensively used in subsequent

papers ([11]), is that a GKC can be represented by means of a planar graph satisfying characteristic topological conditions. On the basis of such properties, many methods for the computerized synthesis of different types of GKC have been developed. The result of the methods is often a large set of graph adjacency lists. Each list represents the graph of a synthesized GKC. The subsequent mapping from the graph to the gear train drawing is executed manually. However, there have been some attempts to carry out this phase also through the help of the computer. Most of the studies on the structural synthesis of mechanisms concentrate only on the completeness of the enumerated set. The practical question of screening among the large number of synthesized geared chains is usually left aside.
For some applications a partial screening can be done taking into account only topological properties. However, at a certain point, the screening process will require, after assigning numerical values to gear ratios and link inertias, kinematic and dynamic analyses.

The results of these analyses will decide for the *best* gear structure.
Because of the large number of geared mechanisms to be analyzed, it is impractical to carry out manually both kinematic and dynamic analyses. Thus, one of the motivations for this work is the casting of an algorithm whose main input data are structured similarly to the common output of computerized synthesis methods (*i.e.* adjacency lists of graphs of GKCs).
An accurate dynamic analysis of epicyclic gear trains requires the consideration of the bodies inertia and the power flowing through each gear pair.

Theoretical analyses and experience demonstrate that the circulating power in an epicyclic gear train (EGT) may be several times the input power. Therefore, for the purpose of avoiding mechanical failures, this type of analysis is recommended at the design stage.

Under the hypotheses of negligible inertia forces, the author of this paper already proposed a systematic algorithm for power flow analysis of planetary spur-gear trains[7, 8]. An implementation in Maple language of such an algorithm is discussed in reference[9].

This paper, after a short introduction of the fundamental topological properties of GKCs, describes:

1. an algorithm for deducing, in symbolic form, the equations of motion of a spur gear EGT;

2. a method for stability analysis of 2 d.o.f. spur-gear EGT.

The systematic nature of the approach adopted in both cases is typical of multibody techniques. This allowed an implementation of the methods within a computer algebra system.

As far as the first method is concerned, with respect to the previous approach, there are the following main differences:

- Power-flow analysis takes into account also inertia forces.
- The meshing forces are explicitly computed.
- The overall inertia of the gear train is estimated.
- The new Maple implementation makes use of the built in procedures of the `networks` package. Thus the length of the code is considerably shortened.

Since methods specifically casted for power-flow analysis in epicyclic gear trains assume constant velocity (e.g. [7, 8, 10]), the first item seems to be novel. The proposed method is based on the joint use of topological properties of GKCs [6, 12] and of a multibody formalism. The structure of input of data makes the computer program developed particularly apt for:

- fully automatic kinematic, power-flow and dynamic analysis of gear trains whose kinematic structure has been obtained through computerized enumeration (e.g.[11]);
- analysis of automatic transmissions gear trains for a given clutching sequence;
- gear train inertia calculation.

The present approach is valid only for those GKCs where the unlimited rotatability of the links is not achieved by means of special proportions.

2 Review of basic topological properties of epicyclic gear trains

The correspondence between mechanisms and graphs maps, respectively, the links and the kinematic pairs of a mechanism into vertices and edges of a graph.

Each edge is labelled according to the nature of the kinematic pair (e.g. G: gear pair, R: revolute pair). Moreover, the level in space of each axis is identified by means of letters $(a, b, c, \ldots)$. Thus, two revolute pairs sharing the same axis level a will be represented with two edges labeled both with

$R(a)$. The Figure 1 gives an example of a graph representing the kinematic structure of an EGT.
Among the properties characterizing the graph $\mathcal{G}$ of a GKC, those that will be used in the following are listed below [6]:

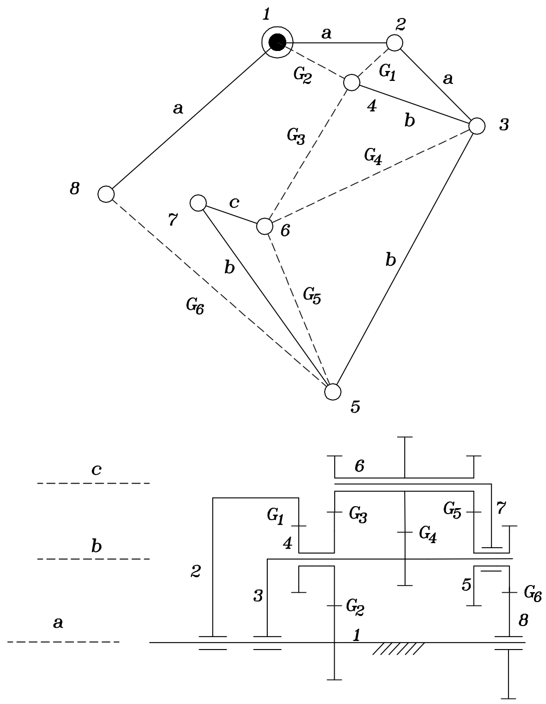

Figure 1: Correspondence between EGTs and graphs.

1. The gear train obeys the Gruebler-Kutzbach equation for the computation of the degree-of-freedom F.

2. The graph is planar.

3. The deletion of all geared edges of $\mathcal{G}$ gives a tree graph or a spanning tree.

4. The number L_{ind} of independent circuits of $\mathcal{G}$ is equal to the number j_G of geared edges.

5. There is a circuit, called *fundamental circuit*, associated to each geared edge. In a fundamental circuit there is only one vertex, denoted as

transfer vertex, dividing revolute edges having the same level (say a) from revolute edges all at a different level (say b). The mechanical counterpart of a transfer vertex is the *gear carrier*.
A *displacement graph* is obtained from the graph of the chain by substituting all the existing turning edges with edges connecting directly the geared edges forming a gear pair with their transfer vertices. The level of the new edges will be the same of those deleted. An elementary gear train is composed of only two gears, say i and j, and one gear carrier k. It is well known that an epicyclic gear train is composed of elementary gear trains.

6. There cannot be within the GKC locked structures or with fractionated mobility. Thus, if Δl links and Δj joints, of which Δj_G are geared, form a kinematic chain which is a part of the entire GKC, then

$$F > 3\Delta l - 2\Delta j + \Delta j_G - 3 \ . \tag{1}$$

2.1 How to find the transfer vertex

It is assumed that the kinematic structure of the GKC is defined by an adjacency list. To find the transfer vertex associated with each gear pair one can adopt the following procedure:

1. Delete all the j_G geared edges from $\mathcal{G}$
2. Add the m-th geared edge
3. Find the links i and j connected by the m-th geared pair
4. Find the set $\{S\}$ of edges of the fundamental circuit associated with the m-th geared pair and delete from this set the geared edge.
5. Split $\{S\}$ into the sets $\{S_a\}$ and $\{S_b\}$ containing only edges labelled with a and b, respectively.
6. The transfer vertex is obtained as the intersection of the two sets of the vertices connected, respectively, by the edges contained in $\{S_a\}$ and $\{S_b\}$.
7. Delete the m-th geared edge and repeat the previous steps for all the remaining geared edges of $\mathcal{G}$

All the described steps can be easily implemented making use of the built-in procedures of the `networks` package of Maple. For the purposes of subsequent analyses it will be useful to compile a look-up table such as Table 1, whose entries refer to the EGT shown in Figure 1.

Fundamental circuit m	Gear i	Gear j	Transfer vertex k	Gear ratio τ_m	Turning pairs level
#1	2	4	3	τ_1	a, b
#2	1	4	3	τ_2	a, b
#3	3	5	4	τ_3	b, c
#4	4	6	7	τ_4	b, c
#5	5	6	7	τ_5	b, c
#6	5	8	3	τ_6	a, b

Table 1: Fundamental circuits look-up table

3 KINEMATIC ANALYSIS

When the compilation of the look-up table is completed, the kinematic analysis of even the most complex EGT become straightforward. Let us denote with ω_i, ω_j and ω_k the absolute angular velocities of the wheels and of the gear carrier, respectively and with τ_m the gear ratio of the m^{th} . Then, the Willis' equation

$$\psi_m \equiv \omega_i - \tau_m \omega_j + (\tau_m - 1)\, \omega_k = 0 \ , \quad (m = 1, 2, \ldots, j_G) \tag{2}$$

can be particularized for each fundamental circuit. To make determinate the system of L_{ind} equations, one has to set equal to zero the angular speed of the ground link and to specify the one of the input link.

3 Kinetic energy of all links

The kinetic energy of all the links of a gear train cannot be deduced only on the basis of the informations embedded in an adjacent links list. In fact, the mapping from the graph to the functional scheme of a corresponding gear train is not unique. To avoid the presence of kinematically *redundant* [1] links, the mentioned list does not contain any hint regarding the number of satellites. This number may vary from 2 up to 12. Examples of EGTs without and with kinematically redundant links are shown in Figure 2. Thus the automatic computation of the kinetic energy, besides the links adjacency list, requires:

[1] In this paper, with the term redundant are denoted those links whose removal does not alter the d.o.f. or the kinematics of the EGT.

- the number of actual gear wheels rigidly connected to a given link[2];
- the number of satellites for each floating gear-wheel;
- the distances δ_{ab}, $\delta_{bc}\dots$ between adjacent rotating axes

The strategy adopted is to compute simbolically the kinetic energy for the most general case and then supply the informations listed above. The kinetic energy of the link l_n, coaxial with the ground link, is

$$K_{l_n} = \frac{1}{2} I_{l_n} \omega_{l_n}^2 \ . \tag{3}$$

For the computation of the kinetic energy of a generic floating link l_n, the

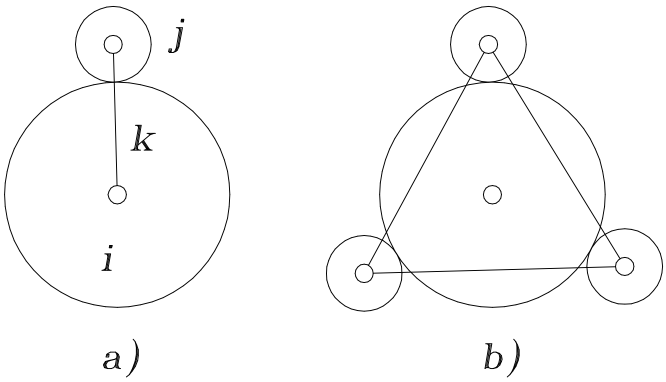

Figure 2: EGTs without (a) and with kinematically redundant links (b).

following approach can be adopted:

1. Delete all the geared edges from $\mathcal{G}$ and form the tree graph $\mathcal{G}_T$ whose root is the frame link f.
2. Within $\mathcal{G}_T$ identify the unique path from f to vertex l_n and form two ordered lists, $L_v = \{f, l_1, l_2, \dots, l_n\}$ and $L_e = \{fl_1\,(a)\,, l_1l_2\,(a)\,, l_2l_m\,(b)\dots,$ $l_{s-1}l_s\,(x)\,, l_sl_n\,(y)\}$, of vertices and edges which form such a path. The label within brackets denote the level of the turning edge. The sequence of vertices and edges goes from f to l_n.

[2]For default the program assumes that there is a distinct gear wheel for each geared pair. However, this is not always the case. The Figure 3 shows how from the same graph can be obtained two different functional representations.

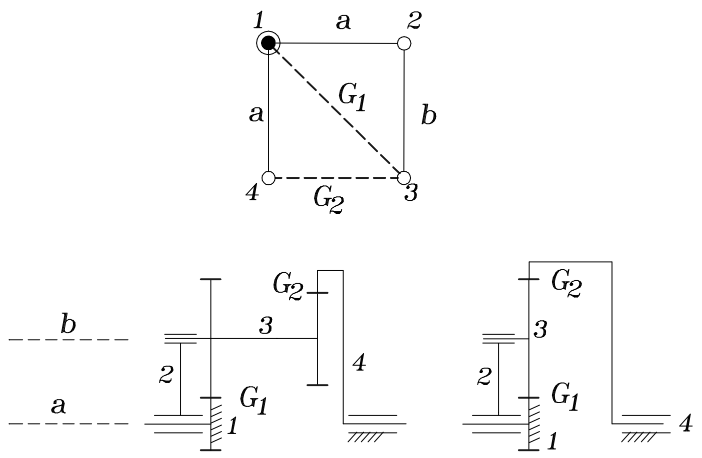

Figure 3: EGTs with identical kinematic structure, but different functional representation.

3. The level of the first edge will be necessarily the lowest one (say a). The kinetic energy of link l_n is computed by means of the expression

$$K_{l_n} = \frac{1}{2} I_{l_n} \omega_{l_n}^2 + \frac{m_i}{2} \|\vec{\omega}_{l_1} \times \vec{\delta}_{aa} + \vec{\omega}_{l_2} \times \vec{\delta}_{ab} + \ldots + \vec{\omega}_{l_s} \times \vec{\delta}_{xy}\|^2 \ , \quad (4)$$

where all the vectors $\vec{\delta}$ with repeated literal subscripts, ($e.g.$ δ_{aa}) must be set equal to zero.

The procedure is better explained by the following example. With reference to the graph tree shown in 4, the sets $\{L_v\}$ and $\{L_e\}$ are, $\{1, 2, 3, 5, 7, 6\}$, $\{12(a), 23(a),\ 35(b), 57(b), 76(c)\}$, respectively. Thus, the kinetic of link 6 is

$$\begin{aligned} K_6 &= \frac{I_6}{2}\omega_6^2 + \frac{m_6}{2}\|\vec{\omega_2} \times \vec{\delta}_{aa} + \vec{\omega_3} \times \vec{\delta}_{ab} + \vec{\omega_5} \times \vec{\delta}_{bb} + \vec{\omega_7} \times \vec{\delta_{bc}}\|^2 \\ &= \frac{I_6}{2}\omega_6^2 + \frac{m_6}{2}\|\vec{\omega_3} \times \vec{\delta}_{ab} + \vec{\omega_7} \times \vec{\delta}_{bc}\|^2 \end{aligned}$$

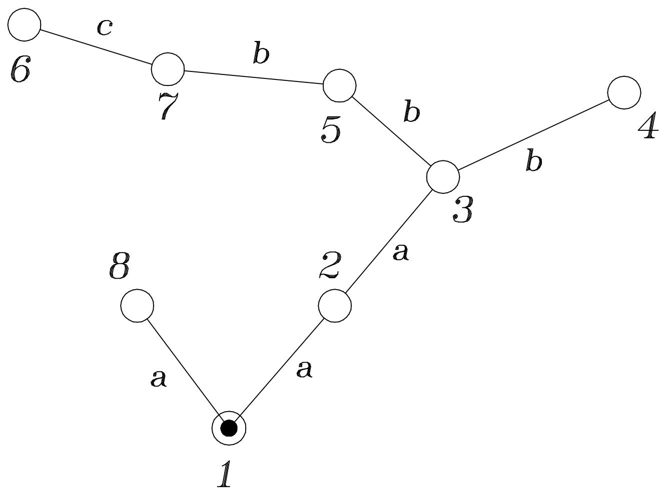

Figure 4: Tree graph for computing the kinetic energy of the links.

Once the kinetic energy K_i of the generic i^{th} link is known, the extended Lagrangian function can be formed[3]

$$L = \sum_{i=2}^{l} K_i + \sum_{m=1}^{j_G} \lambda_m \psi_m \ , \tag{5}$$

where the λs are the Lagrange's multipliers.

By applying the equations of Lagrange, one obtains the equations of motion in the following form

$$[M]\{\dot{\omega}\} + [\psi_\omega]^T \{\lambda\} = \{Q\} \ , \tag{6}$$

where

- $\{\dot{\omega}\}$ is the vector of angular accelerations;
- $[M]$ is mass matrix;
- $\{Q\} = \{T_d, 0, \ldots, 0, T_r\}$ is the vector of externally applied generalized forces.

[3]The link frame is denoted with $i = 1$. Moreover, it is assumed that the input and output angular velocities are, respectively, the first and last elements of vector $\{\omega\}$. The procedure outlined in the following can be generalized to multi d.o.f. gear drives. However, this algebraic treatment is limited to one d.o.f. gear trains.

The vector $\{\mathcal{T}\}$ of torques due to meshing and reaction forces is represented by

$$\{\mathcal{T}\} = [\psi_\omega]^T \{\lambda\} \ . \tag{7}$$

4 Reaction and meshing forces

The solution of the inverse dynamics problem assumes as data:

- the kinematic structure of the EGT;
- the resisting torque T_r;
- the required angular acceleration α_{out} of the output link;
- the link inertias;
- the gear ratios and the primitive radius of each gear.

The unknowns are the reaction and meshing forces and the input torque T_d.

Let $[\psi_\omega]$ be the Jacobian matrix associated with the system of equations (2). Moreover, let $[\Psi_\omega]$ the square matrix obtained adding an extra row to $[\psi_\omega]$. The row has all entries equal to zero, except the first element relative to the input link, which is set equal to the unit value.

The vector $\{\dot{\omega}\}$ can be computed by solving the system

$$[\Psi_\omega]\{\dot{\omega}\} = \{\alpha\} \ , \tag{8}$$

where, denoted with α_{out} the prescribed angular acceleration of the output link, is $\{\alpha\} = \{0, 0, \ldots, \alpha_{out}\}$. Finally, introduced the vector

$$\{\lambda^*\} = \left\{ \begin{array}{c} \lambda \\ \lambda^\star_{j_G+1} \end{array} \right\} \tag{9}$$

with $j_G + 1$ elements, from (6) and (8) one obtains

$$\{\lambda^\star\} = \left[\Psi^T_\omega\right]^{-1} \Big(\{Q\} - [M]\,[\Psi_\omega]^{-1}\{\alpha\}\Big) \ . \tag{10}$$

By solving the equation

$$\lambda^\star_{j_G+1} = 0 \ ,$$

the value of T_d is computed.

If the subscript s denotes a wheel of the m^{th} gear pair, then the meshing force F_{sm} at the m^{th} gear pair is given by

$$F_{sm} = \frac{\lambda_m \psi^T_\omega [s, m]}{r_{sm}} \ , \quad (m = 1, \ldots, j_G) \ . \tag{11}$$

5 Stability analysis of a 2 d.o.f. EGT

Some type of analysis do not require the explicit computation of meshing forces.
For instance, a model for the forward dynamic analysis of a two d.o.f. EGT can be described by the following differential equations system

$$[\mathcal{M}] \left\{ \begin{array}{c} \dot{\omega}_{d1} \\ \dot{\omega}_{d2} \end{array} \right\} = \left\{ \begin{array}{c} T_{d1} - aT_r \\ T_{d2} - bT_r \end{array} \right\} , \tag{12}$$

where

- $[\mathcal{M}]$ is reduced mass matrix,
- T_{d1}, T_{d2} and T_r are respectively the driving and resisting torques,
- $\dot{\omega}_{d1}$, $\dot{\omega}_{d2}$ are the absolute angular accelerations at the driving shafts,
- a and b are constants function of gear ratios.

Under certain working conditions, it can be shown that the behavior of the system is not stable.
For this purpose, let us assume that the motor speed-torques curves are linearized by means of the following equations

$$\begin{aligned} T_{d1} &= T_{d1}^o + K_{d1}\left(\omega_{d1} - \omega_{d1}^o\right) \ , & (13) \\ T_{d2} &= T_{d2}^o + K_{d2}\left(\omega_{d2} - \omega_{d2}^o\right) \ , \ , & (14) \end{aligned}$$

where the upperscript ^{o} refers to the values of variables at steady state condition. Similarly, the resisting torque-speed curve can be also described by means of the linear equation

$$T_r = T_r^o + K_r\left(\omega_r - \omega_r^o\right) \ , \tag{15}$$

where ω_r denotes the angular speed of the output shaft. When (13), (14) and (15) are substituted into (12), after taking into account the existing linear relationship between ω_r and the input angular velocities ω_{d1} and ω_{d2} of the two driving shafts, with obvious nomenclature, one obtains:

$$[\mathcal{M}] \left\{ \begin{array}{c} \Delta\dot{\omega}_{d1} \\ \Delta\dot{\omega}_{d2} \end{array} \right\} + [\mathcal{K}] \left\{ \begin{array}{c} \Delta\omega_{d1} \\ \Delta\omega_{d2} \end{array} \right\} = \left\{ \begin{array}{c} T_{d1}^o - aT_r^o \\ T_{d2}^o - bT_r^o \end{array} \right\} , \tag{16}$$

with $[\mathcal{K}]$ symmetric matrix. The solution of the homogeneous o.d.e. associated with (16) will have the following algebraic form

$$\left\{ \begin{array}{c} \Delta\omega_{d1} \\ \Delta\omega_{d2} \end{array} \right\} = \{X\}\,\mathrm{e}^{\lambda t} \ , \tag{17}$$

where λ is the solution of the eigenvalue problem

$$[\lambda\mathcal{M} + \mathcal{K}]\,\{X\} = \{0\} \ . \tag{18}$$

It is self evident that positive values of λ's will reveal an unstable behavior of the system. Since this type of analysis requires a preliminary computation of $[\mathcal{M}]$ and $[\mathcal{K}]$, in the section will be described a procedure for the computation of such matrices.
Let

$$\phi_1 \equiv \dot{p}_1 - \omega_{d1} = 0 \ , \tag{19}$$
$$\phi_2 \equiv \dot{p}_2 - \omega_{d2} = 0 \ , \tag{20}$$
$$\tag{21}$$

and

$$\Gamma(\omega, \dot{p}) = \left\{ \begin{array}{c} \{\psi(\omega)\} \\ \{\phi(\omega, \dot{p})\} \end{array} \right\} = \{0\} \tag{22}$$

the vector of constraints.
Once the matrix

$$[V] = -\left[\Gamma_\omega\right]^{-1}\left[\Gamma_{\dot{p}}\right] \ , \tag{23}$$

is introduced, the following orthogonality relationship

$$[\psi_\omega]\,[V] = [0] \tag{24}$$

can be demonstrated [14, 15].
Making use of this condition one obtains

$$[\mathcal{M}] = [V]^T\,[M]\,[V] \tag{25}$$

and

$$[\mathcal{K}] = [V]^T\,\{T\} \ , \tag{26}$$

where

$$\{T\} = \left\{ \begin{array}{c} \dots \\ K_{d1}\left(\omega_{d1} - \omega_{d1}^o\right) \\ \dots \\ K_{d2}\left(\omega_{d2} - \omega_{d2}^o\right) \\ \dots \\ K_r\left(\omega_r - \omega_r^o\right) \\ \dots \end{array} \right\} \tag{27}$$

is the vector of externally applied torques [4]

6 Examples

6.1 Torque and meshing forces analysis

The procedure described in this paper has been used for the complete inverse dynamic analysis of the EGT shown in Figure 3. For this purpose, let:

- $\tau_1 = -\frac{r_{31}}{r_{11}}$ and $\tau_2 = -\frac{r_{42}}{r_{32}}$;
- T_r the resisting torque applied to link 4;
- $\dot{\omega}_i$ the angular acceleration of link i (i=2,3,4);
- n_{s3} number of satellites of body 3;

The equations for kinematic analysis are

$$\begin{aligned} \psi_1 &\equiv \tau_1\omega_3 + (1-\tau_1)\,\omega_2 = 0 \\ \psi_2 &\equiv \omega_3 - \tau_2\omega_4 + (\tau_2 - 1)\,\omega_2 = 0 \ . \end{aligned}$$

Thus, the Jacobian matrices to be used in equations (10) are defined as follows

$$[\psi_\omega] = \begin{bmatrix} \tau_1 & 1-\tau_1 & 0 \\ 1 & \tau_2 - 1 & -\tau_2 \end{bmatrix} ,$$

$$[\Psi_\omega] = \begin{bmatrix} \tau_1 & 1-\tau_1 & 0 \\ 1 & \tau_2 - 1 & -\tau_2 \\ 0 & 0 & 1 \end{bmatrix} .$$

The elements of vector $\{\dot{\omega}\} = \{\dot{\omega}_2, \dot{\omega}_3, \dot{\omega}_4\}^T$ follow from kinematic analysis. Since $\alpha_{out} = \dot{\omega}_4$ is in this case prescribed, we have

$$\begin{aligned} \dot{\omega}_2 &= \frac{\tau_1\tau_2}{\tau_1\tau_2 - 1}\alpha_{out} \ , \\ \dot{\omega}_3 &= \frac{(\tau_1 - 1)\,\tau_2}{\tau_1\tau_2 - 1}\alpha_{out} \ . \end{aligned}$$

Moreover, since the kinetic energy of the entire EGT is

$$\begin{aligned} K &= K_2 + K_3 + K_4 \\ &= \frac{1}{2}\left(I_2 + n_{s3}m_3\delta_{ab}^2\right)\omega_2^2 \\ &\quad + \frac{1}{2}n_{s3}I_3\omega_3^2 + \frac{1}{2}I_4\omega_4^2 \ , \end{aligned}$$

[4] At this stage of the analysis procedure the values of T_{d1}^o, T_{d2}^o and T_r^o can be omitted.

the mass matrix has the following expression

$$[M] = \begin{bmatrix} I_2 + m_3 n_{s3} \delta_{ab}^2 & 0 & 0 \\ 0 & I_3 n_{s3} & 0 \\ 0 & 0 & I_4 \end{bmatrix} .$$

The vector of externally applied torques is

$$\{Q\} = \{T_d, 0, T_r\}^T \ .$$

The development of the matrix expression (10) gives

$$\{\lambda^\star\} = \{\lambda_1, \lambda_2, \lambda_3\}^T \ .$$

When the last component $\lambda_3^\star$ is set equal to zero, one obtains the expression of the unknown driving torque

$$\begin{aligned} T_d &= T_r \left(1 - \frac{1}{\tau_1 \tau_2}\right) + \alpha_{out} \left[\left(I_2 + m_3 \delta_{ab}^2 n_{s3}\right) \frac{\tau_1 \tau_2}{\tau_1 \tau_2 - 1} \right. \\ & \left. + I_3 n_{s3} \frac{\tau_2 (\tau_1 - 1)^2}{\tau_1 (\tau_1 \tau_2 - 1)} + I_4 \left(1 - \frac{1}{\tau_1 \tau_2}\right) \right] . \end{aligned}$$

The computed meshing forces are

- Gear pair G_1

$$\begin{aligned} F_{31} &= \frac{\lambda_1 \psi_\omega^T [3,1]}{r_{31}} = \frac{1}{r_{31} \tau_2} (T_r - I_4 \alpha_{out}) - \frac{I_3 n_{s3} \tau_2 (\tau_1 - 1)}{r_{31} (\tau_1 \tau_2 - 1)} \alpha_{out} \\ F_{41} &= 0 \end{aligned}$$

- Gear pair G_2

$$\begin{aligned} F_{32} &= \frac{\lambda_2 \psi_\omega^T [3,2]}{r_{32}} = -\frac{T_r - I_4 \alpha_{out}}{r_{32} \tau_2} \\ F_{42} &= \frac{\lambda_2 \psi_\omega^T [4,2]}{r_{42}} = \frac{(T_r - I_4 \alpha_{out})}{r_{42}} \end{aligned}$$

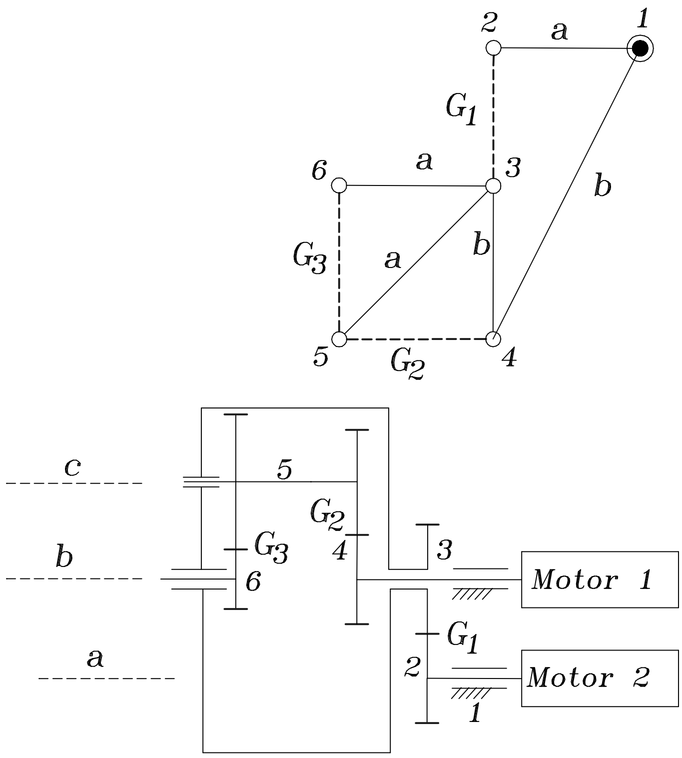

Figure 5: Functional representation of a 2 d.o.f. rotary drive mechanism.

6.2 Stability analysis

Let us assume that is required to deduce the matrices $[\mathcal{M}]$ and $[\mathcal{K}]$ for the 2 d.o.f. gear train shown in Figure 5. This EGT has been developed at the Ames Research Center, Moffet Field (CA).

Fundamental circuit	Gear i	Gear j	Transfer vertex k	Gear ratio τ	Turning pairs level
#1	2	3	1	τ_1	a, b
#2	5	4	3	τ_2	b, c
#3	5	6	3	τ_3	a, c

Table 2: Fundamental circuits look-up table (see Figure 5)

From Table 2 one obtains the following system of equations

$$\{\psi\} \equiv [\psi_\omega]\,\{\omega\} = \begin{bmatrix} 1 & -\tau_1 & 0 & 0 & 0 \\ 0 & \tau_2 - 1 & -\tau_2 & 1 & 0 \\ 0 & \tau_3 - 1 & 0 & 1 & -\tau_3 \end{bmatrix} \left\{ \begin{array}{c} \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \\ \omega_6 \end{array} \right\} = \{0\} \ .$$

Solving this set of equations w.r.t. ω_3, ω_5 and ω_6 one obtains

$$\begin{aligned} \omega_3 &= \frac{\omega_2}{\tau_1} \ , \\ \omega_5 &= \frac{1-\tau_2}{\tau_1}\omega_2 + \tau_2\omega_4 \ , \\ \omega_6 &= \frac{\tau_3-\tau_2}{\tau_1\tau_3}\omega_2 + \frac{\tau_2}{\tau_3}\omega_4 \ . \end{aligned}$$

Since

$$\begin{aligned} \phi_1 &\equiv \dot{p}_1 - \omega_2 = 0 \ , \\ \phi_2 &\equiv \dot{p}_2 - \omega_4 = 0 \ , \end{aligned}$$

from (23) the matrix

$$[V] = \begin{bmatrix} 1 & 0 \\ \frac{1}{\tau_1} & 0 \\ 0 & 1 \\ \frac{1-\tau_2}{\tau_1} & \tau_2 \\ \frac{\tau_3-\tau_2}{\tau_1\tau_3} & \frac{\tau_2}{\tau_3} \end{bmatrix}$$

is deduced.
Finally, from the application of (25) and (26) one obtains, respectively,

$$
\begin{aligned}
\mathcal{M}_{11} &= I_2 + \frac{I_3 + m_5\delta_{bc}^2}{\tau_1^2} + \left(\frac{\tau_2 - 1}{\tau_1}\right)^2 I_5 + \left(\frac{\tau_2 - \tau_3}{\tau_1\tau_3}\right)^2 I_6 \ , \\
\mathcal{M}_{12} &= \mathcal{M}_{21} = \frac{(1-\tau_2)\,\tau_2}{\tau_1} I_5 + \frac{(\tau_3 - \tau_2)\,\tau_2}{\tau_1\tau_3^2} I_6 \ , \\
\mathcal{M}_{22} &= I_4 + I_5\tau_2^2 + \frac{\tau_2^2}{\tau_3^2} I_6 \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{K}_{11} &= K_{d1} + \left(\frac{\tau_2 - \tau_3}{\tau_1\tau_3}\right)^2 K_r \ , \\
\mathcal{K}_{12} &= \mathcal{K}_{21} = \frac{\tau_2\,(\tau_3 - \tau_2)}{\tau_1\tau_3^2} K_r \ , \\
\mathcal{K}_{22} &= K_{d2} + \frac{\tau_2^2}{\tau_3^2} K_r \ .
\end{aligned}
$$

7 Conclusions

It has been proposed a method of inverse dynamic analysis of epicyclic gear trains (EGT). The method makes use of the topological properties of the graphs representing EGTs. A computer program has been developed in Maple V Release 4. A characteristic feature of such program is the format of input data. In particular, the kinematic structure of the mechanism is described by means of the verteices adjacency list of the corresponding graph. This is the same format usually adopted by enumeration techniques of EGTs. This allows to automatically generate not only a non isomorphic kinematic structure, but also the equations required for its inverse dynamic analysis.
Another novelty of this approach is the possibility to evaluate the contribution of inertia on meshing forces. Often the EGT is analyzed under the hypotheses of constant angular velocities (e.g. [8, 10, 13]). However, under transient working conditions this hypothesis needs to be removed. The method allows also to compute the equivalent inertia of the entire mechanism reduced to the input shaft. This calculation is required for the estimation of torsional frequencies of the transmission shaft. Finally it has been presented a multibody approach to stability analysis of 2 d.o.f. EGTs.

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