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KINEMATIC ANALYSIS OF THE ANTIKYTHERA GEAR MECHANISM BY MEANS OF GRAPH THEORY

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ABSTRACT

The Antikythera mechanism is one of the most extraordinary examples of the degree of sophistication reached by the Hellenistic culture in the field of mechanism design. This calendar mechanism not only witness the degree of astronomic knowledge of the greeks, but also their capability to conceive and build marvelous mechanical devices. The mechanism is the first documented example of differential gear arrangement in western culture.

The paper gives first a description of the kinematic structure of the mechanism in terms by means of a graph, then a full kinematic analysis of this complex gear train is performed.

The results of our kinematic analysis, only partially coincident with those obtained by De Solla Price, match very closely the discoveries made by the Greek astronomer Hipparchus about the duration of the lunar months.

There is also a conjecture on the kinematic structure and gear dimensions of some of the missing parts.

In the Appendix is reported the listing of the Maple program developed for the kinematic analysis.

THE HISTORY OF THE DISCOVERY

The discovery of this archaeological treasure was made by sponge-fishers at the beginning of 1900. Because of a storm they

decided to anchor their boat about one mile east of a port known locally as Pinakakia. The diver Elias Stadiatis, at a the depth of 42 meters, found the wreck of a ship. By september 1901, the sponge divers, contracted for the job by the Greek Archaeological Society, recovered most of the objects on the wreck surface. The principal works have now a place in the Archeological Museum of Athens.

Likely, the ship was a commercial boat vessel that, after leaving a port in central coast of Asia minor, or some adjacent Greek island, was heading westwards through the channel on its way to Italy. Archaeologists agree that the date of the ship wreck was about 80-50 B.C.

DESCRIPTION OF THE MECHANISM

The achievements of the ancient greek culture in the field of philosophy, geometry and mathematics are well known. They are the pillars on which rest our modern civilization. These achievements in merely speculative fields may give the impression of a greek culture not very close to the daily needs of people. On the contrary, as stated in (Dimarogonas, 1991):

“Rapid advancement in natural sciences was followed by systematic attempts to organize the knowledge in engineering and, in particular, in Mechanical Design, developing it beyond the level of a mere craft.”

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The statement is very well supported by historical evidence. In fact, although most of the works have been lost, some parts of the treatises written by the ancient greek engineers are nowadays available in Arab translations. Historic reviews of the advancements obtained by the greek culture are available from many sources (*e.g.* (Clagett, 1955; Farrington, 1953)) and outside the purpose of this paper.

However, this culture made a clear distinction, forgotten in the western world until the Renaissance (1400 a.C.), between the craftsman and the designer of mechanical devices.

For instance, the following sentence, whose modernity can be recognized,

“The mechanics (more precisely: the mechanical engineers, OI MHXANIKOI) of Heron’s school told us that Mechanicals consist of a theoretical and a practical part. The theoretical part includes the natural sciences while the practical part consists of the engineering disciplines. They postulate that a necessary condition for an able inventor of mechanical devices is a solid background in natural sciences and practical skills.....”

is attributed to Pappos of Alexandria (400 b.C.).

The Antikythera mechanism is the earliest document of the established tradition in the Hellenistic world of manufacturing mathematical gearing. This tradition was continued by the Bizantine civilization and, later, by the Islamic one. As stated before, this tradition distinguished the designer from the manufacturer. In the case of the Antikythera mechanism, the designer had necessarily a deep knowledge of astronomy and of kinematics. Robert Willis, an english professor of engineering at university of Cambridge (1837), is usually credited for the formula relating the angular speeds of gears in differentials. However, the Antikythera mechanism witness that its designer mastered that formula more than twenty centuries before R. Willis.

In the Science Museum in London are exposed examples of Bizantine and islamic geared calendars that, although much less sophisticated than the Antikythera mechanism (only ordinary gear trains were used), could share with it some of the functions.

Cicero (106-43 B.C.) simply reports the existence of a mechanical model, build by Archimedes (287-212 B.C.), for the reproduction of the motion of heavenly bodies. However, no hint is given about its structure. One may conjecture that the Antikythera mechanism was the one mentioned by Cicero. Unfortunately, the identity of the inventor and the locus of invention are still lost in time.

The historical importance of the device under analysis is well explained by the following statement (De Solla, 1974):

“With the Antikythera mechanism, whatever its function, we are evidently concerned with a rather different phenomenon of High Technology. This is the name we

give to those especially sophisticated crafts and manufactures that are in some ways intimately associated with the sciences, drawing on them for theories, giving to them the instruments and the techniques that enable men to observe and experiment and increase both knowledge and technical competence.”

The presence of a differential and of a wheel with double grinding demonstrate also the technological capabilities of the manufacturer. It is useful to mention that western world will wait almost fourteen centuries before a documented use of differential gears.

The Antikythera mechanism was contained in a frame box made of wood. The main parts are:

- Front and back doors on which the user’s instructions were inscribed.
- The input crank wheel (link 2).
- Sun and Moon indicator plates (links 3 and 5, respectively).

The wheels are obtained from bronze sheets having a thickness from 2.0 to 2.3 millimeters.

The purpose of the device is not completely known. It could have been used for different practical purposes such as (Field and Wright, 1985):

- conversion from the lunar calendar (used by peasants) to the Julian calendar (used by upper class);
- prediction of the moon position and size;
- position of sun and moon in the zodiac.

The definitive study on the Antikythera mechanism can be found in (De Solla, 1959; De Solla, 1974).

THE KINEMATIC STRUCTURE

The planar embodiement of the gear train proposed by (De Solla, 1974) is shown in Figure 1. The authors added the main dimensions of the wheels in centimeters.

Graphs are not only useful tools for the description of the kinematic structure of a gear train, but provide also a framework for the use of systematic methodologies for their kinematic analysis (Buchsbaum and Freudenstein, 1970; Freudenstein, 1971; Yang and Freudenstein, 1972).

With reference to the link numbering scheme of Figure 2, the graph of the Antikythera mechanism is shown in Figure 3.

Each gear (revolute) pair has been denoted with the letter G (R) and an arab number.

The question marks in the drawing refer to the missing parts. It is clear that the actual structure of such parts can only be conjectured.

There are 20 links (frame included), 19 revolute pairs, 18 gear pairs, thus the degree-of-freedom of the gear train is

$$F = j_R - j_G ,$$

Table 1. Characteristics of the fundamental circuits (1-6)

	G1	G2	G3	G4	G5	G6
Body i	2	2	4	4	4	5
Body j	3	4	10	6	15	11
Body k	1	1	1	1	1	1
Z_i	Z_2	Z_{a2}	Z_{a4}	Z_{a4}	Z_{b4}	Z_{a5}
Z_j	Z_3	Z_4	Z_{10}	Z_6	Z_{15}	Z_{a11}

Table 2. Characteristics of the fundamental circuits (7-12)

	G7	G8	G9	G10	G11	G12
Body i	5	6	7	8	10	12
Body j	12	7	8	9	11	13
Body k	1	1	1	1	1	20
Z_i	Z_{a5}	Z_{a6}	Z_{a7}	Z_8	Z_{a10}	Z_{a12}
Z_j	Z_{12}	Z_7	Z_8	Z_9	Z_{11}	Z_{13}

Table 3. Characteristics of the fundamental circuits (8-18)

	G13	G14	G15	G16	G17	G18
Body i	13	16	14	16	17	18
Body j	14	20	15	17	18	19
Body k	20	1	20	1	1	1
Z_i	Z_{13}	Z_{16}	Z_{a14}	Z_{a16}	Z_{a17}	Z_{a18}
Z_j	Z_{14}	Z_{a20}	Z_{a15}	Z_{17}	Z_{18}	Z_{19}

The number of gear teeth are reported in the following list:

$$\begin{aligned}
 z_2=45, \quad z_{a2}=45, \quad z_3=225, \quad z_4=225, \quad z_{a4}=64, \\
 z_{b4}=32, \quad z_{a5}=32, \quad z_6=36, \quad z_{a6}=54, \quad z_7=96, \\
 z_{a7}=16, \quad z_8=64, \quad z_9=32, \quad z_{a9}=48, \quad z_{10}=38, \\
 z_{a10}=48, \quad z_{11}=24, \quad z_{a11}=127, \quad z_{12}=32, \quad z_{a12}=32, \\
 z_{13}=64, \quad z_{14}=32, \quad z_{a14}=48, \quad z_{15}=32, \quad z_{a15}=48, \\
 z_{16}=48, \quad z_{a16}=30, \quad z_{17}=60, \quad z_{a17}=20, \quad z_{18}=15, \\
 z_{a18}=15, \quad z_{19}=60, \quad z_{a20}=192.
 \end{aligned}$$

Let us denote with ω_i , ω_j and ω_k the absolute angular velocities of the wheels and of the gear carrier, respectively and with

Table 4. Results of the kinematic analysis

ω_1	ω_2 (input)	ω_3	ω_4	ω_5
0	5	-1	-1	13.3684
ω_6	ω_7	ω_8	ω_9	ω_{10}
1.7777	-1	0.2500	-0.5000	1.6842
ω_{11}	ω_{12}	ω_{13}	ω_{14}	ω_{15}
-3.3684	-13.3684	-2.5921	-13.3684	1
ω_{16}	ω_{17}	ω_{18}	ω_{19}	ω_{20}
24.7368	-12.3684	4.1228	-1.0307	-6.1842

τ_m the gear ratio of the m^{th} . Then, the so called Willis' equation

$$\Psi_m \equiv \omega_i - \tau_m \omega_j + (\tau_m - 1) \omega_k = 0, (m = 1, 2, \dots, j_G) \quad (2)$$

can be particularized for each fundamental circuit. To make determinate the system of L_{ind} equations, one has to set equal to zero the angular speed of the ground link and to specify the one of the input link.

For the purpose of solving a linear system with 18 equations a Maple program has been developed (see the enclosed Maple Worksheet).

By adopting for each gear the number of gear teeth proposed in (De Solla, 1974), one obtains the following results: For the interpretation of the results one must take into account the following definitions:

- The *sidereal month* is the time it takes for the Moon to complete one full orbit of the Earth, measured with respect to the distant stars. In determining a sidereal month the distant stars are assumed fixed relative to Earth.
- The *synodic month* is the time it takes for the Moon to complete one cycle of phases. For example, it is the time it takes the Moon to complete one full cycle of phases. That is, the time between successive new moons.

The duration of these months are reported in Table 5. It is worth to mention that the Greek astronomer Hipparchus, using naked eye observations, estimated in 365.242 days and 29.53058 the duration of solar year and the sidereal month, respectively. However, these discoveries were virtually ignored by calendar makers¹.

¹The calendrical reform of Julius Caesar (46 b.C. established a calendar year of 365.25 days (more than 11 minutes too long). In 1500 a.C. the Julian calendar was about 10 days behind the solar year. The mistake was eliminated with

Table 5. Actual time measurements

Earth	Moon	Moon
(Solar year)	(Synodic month)	(Sidereal month)
365.242199 days	29.53059 days	27.3229 days

Table 6. Missing gear wheels:conjectured teeth numbers

Planet	No. teeth O_1	No. teeth O_2	No. teeth X
Venus	32	50	40
Venus	31	52	43
Venus	32	48	41
Mars	30	50	57
Mars	31	48	53
Jupiter	30	48	28
Jupiter	32	53	29

From the numerical results summarized in Table 4, it is clear that:

- Five turns of the input crank (link 2) correspond to one solar year.
- The gear wheels 3 and 5 represent, respectively, the position of the sun and moon in the zodiac. According to the Antikythera mechanism, the ratio between the solar year and the sidereal month is given by the numerical value of $\omega_5=13.3684$. The actual value is 13.3676.
- Gear wheel 19 rotation is the ratio between the solar year and the synodic month. The actual value is $365.242199/29.5308=12.3681$, whereas the value restituted by the Antikythera mechanism is $\omega_{17}=12.3684$. Derek de Solla Price stated that the ratio restituted by the mechanism is $236/19=12.4210$.
- The gear wheel 9 measured the ratio of the solar year and lunar year. The actual value is $365.242199/(29.53059*12)=1.03068$, whereas the value restituted by the Antikythera mechanism is $\omega_{19}=1.0307$.
- Gear wheel 8 makes one turn each four solar years. This gear seems not engaged with any other. Likely the designer was already aware of the problem of leap years.

The Greek tradition of recording calendrical ratios has been continued by the Arabic culture, as witnessed by the writings of the Persian mathematician and philosopher Al-Biruni (973-1048). However, in this case only astrolabes with ordinary gear trains were described and built. (Field and Wright, 1985)

A CONJECTURE ON THE MISSING PARTS

From the parts recovered, one can reconstruct three dials (one in the front side and two in the rear side). Taking into account the level of astronomical knowledge of Hellenistic astronomers, it seems reasonable conjecturing that the other dials of the mechanism reproduce the motion of Mars, Jupiter and Venus or the *saros*² time.

the calendar reform of Pope Gregory XIII who accepted the length of the year to 365.2422 days, adjusted the leap year rule and eliminated the accumulated 10 extra days. University of Rome "Tor Vergata" is somewhat linked to the calendar reform of Pope Gregory XIII. In fact, the university today owns Villa Mondragone, the beautiful residence in the neighborhood of Rome where the pope officially declared his new reform.

²The eclipse cycle, discovered by the Babylonian astronomers, has a period of 223 synodic months, or 6,585.32 days (18 years and about 11 days).

The missing part of the gear train should mesh with the wheel N having 64 gear teeth and an angular speed of 0.25 turns each solar year.

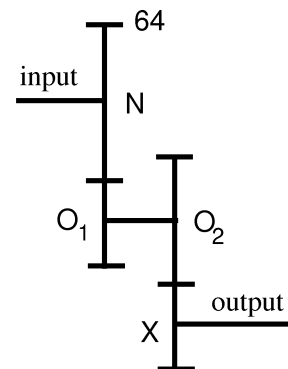


Figure 4. Conjectured gear arrangement of the missing parts

Let us denote with O_1 , O_2 and X the missing gears. If the arrangement of the gear train was the one shown in Figure 4, the Table 6 summarizes the number of gear teeth of the missing wheels³.

If the missing gear train reproduced the *saros* period, then the required overall transmission ratio was $\tau=6585.32/(4*365.2422)=4.5075$. This could have been achieved with an arrangement similar to the one shown in Figure 4.

³These values have been analytically computed by means of the method described in (Di Benedetto, Pennestrì, 1997).

CONCLUSIONS

The kinematic structure and the wheel motion of the Antikythera gear mechanism (see Figure 5) have been systematically analyzed with a modern approach. By means of the computer program developed one can study the influence on the overall output of the number of teeth of any gear wheel. This capability is important since, for some gears, there is still uncertainty on the exact number of teeth. This number has been sometimes estimated from the wheel radius.

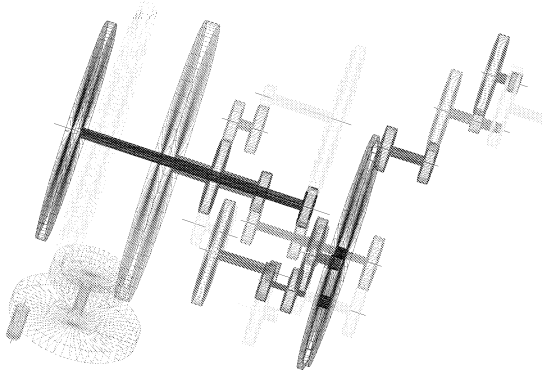


Figure 5. Three dimensional view of the Antikythera mechanisms

In this paper it has been presented a full kinematic analysis of all the gears composing the device. The velocity ratios of the main dial gears are in partial agreement with those reported in (De Solla, 1974). In fact, according to our results, the accuracy of the Antikythera mechanism in computing the duration of the synodic and sidereal months seems higher than the one estimated in (De Solla, 1974) and consistent with the calculations of the Greek astronomer Hipparchus.

Finally a conjecture about some of the missing parts has been proposed.

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Maple V computer program listing

```
# KINEMATIC ANALYSIS OF
# THE ANTIKITHERA MECHANISM
> restart:
# Gear teeth number according to
# de Solla's nomenclature
> Z(H2):=15: Z(M2):=16: Z(G1):=20:
> Z(D1):=24: Z(F2):=30: Z(E1):=32:
> Z(L1):=36: Z(C1):=38: Z(C2):=48:
> Z(A):=48: Z(E5):=48: Z(F1):=48:
> Z(K2):=48: Z(O2):=48: Z(L2):=54:
> Z(G2):=60: Z(H1):=60: Z(I):=60:
> Z(N):=64: Z(B2):=64: Z(M1):=96:
> Z(EK):=96: Z(D2):=127: Z(E3):=192:
> Z(E4):=222: Z(B1):=225:
# Gear teeth number according to
# our nomenclature
> Z2:=45: Za2:=45: Z3:=225: Z4:=225:
> Za4:=64: Zb4:=32: Za5:=32: Z6:=36:
> Za6:=54: Z7:=96: Za7:=16: Z8:=64:
> Z9:=32: Za9:=48: Z10:=38: Za10:=48:
> Z11:=24: Za11:=127: Z12:=32: Za12:=32:
> Z13:=64: Z14:=32: Za14:=48: Z15:=32:
> Za15:=48: Z16:=48: Za16:=30: Z17:=60:
```

```

> Za17:=20: Z18:=60: Za18:=15: Z19:=60:
> Za20:=192:
# Compute velocity ratios
# Link 1 is the frame.
# Link 2 is the input gear.
> n[1]:=0: n[2]:=5:
# Generate the kinematic analysis equations:
> eq1:=(n[2]-n[1])/(n[3]-n[1])+Z3/Z2=0:
> eq2:=(n[2]-n[1])/(n[4]-n[1])+Z4/Za2=0:
> eq3:=(n[4]-n[1])/(n[10]-n[1])+Z10/Za4=0:
> eq4:=(n[4]-n[1])/(n[6]-n[1])+Z6/Za4=0:
> eq5:=(n[4]-n[1])/(n[15]-n[1])+Z15/Zb4=0:
> eq6:=(n[5]-n[1])/(n[11]-n[1])+Za11/Za5=0:
> eq7:=(n[5]-n[1])/(n[12]-n[1])+Z12/Za5=0:
> eq8:=(n[6]-n[1])/(n[7]-n[1])+Z7/Za6=0:
> eq9:=(n[7]-n[1])/(n[8]-n[1])+Z8/Za7=0:
> eq10:=(n[8]-n[1])/(n[9]-n[1])+Z9/Z8=0:
> eq11:=(n[10]-n[1])/(n[11]-n[1])+Z11/Za10=0:
> eq12:=(n[12]-n[20])/(n[13]-n[20])+Z13/Za12=0:
> eq13:=(n[13]-n[20])/(n[14]-n[20])+Z14/Z13=0:
> eq14:=(n[16]-n[1])/(n[20]-n[1])+Za20/Z16=0:
> eq15:=(n[14]-n[20])/(n[15]-n[20])+Za15/Za14=0:
> eq16:=(n[16]-n[1])/(n[17]-n[1])+Z17/Za16=0:
> eq17:=(n[17]-n[1])/(n[18]-n[1])+Z18/Za17=0:
> eq18:=(n[18]-n[1])/(n[19]-n[1])+Z19/Za18=0:
#
> solve({eq1,eq2,eq3,eq4,eq5,eq6,eq7,eq8,
eq9,eq10,eq11,eq12,eq13,eq14,eq15,eq16,
eq17,eq18},{n[3],n[4],n[5],n[6],n[7],n[8],
n[9],n[10],n[11],n[12],n[13],n[14],
n[15],n[16],n[17],
n[18],n[19],n[20]});

```

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	R(2)				R(5)	R(6)	R(7)	R(8)	R(9)	R(10)	R(11)				R(15)	R(16)	R(17)	R(18)	R(19)	
2	R(2)		G1	G2																
3		G1			R(5)															
4		G2			R(5)	G4				G3					G5					
5	R(5)		R(5)	R(5)							G6	G7								
6	R(6)			G4			G8													
7	R(7)							G8		G9										
8	R(8)						G9		G10											
9	R(9)							G10												
10	R(10)			G3							G11									
11	R(11)					G6				G11										
12					G7								G12		R(15)					
13												G12		G13						R(13)
14													G13		G15					R(14)
15	R(15)			G5								R(15)		G15						R(15)
16	R(16)															G16				G14
17	R(17)															G16	G17			
18	R(18)																G17	G19		
19	R(19)																	G18		
20													R(13)	R(14)	R(15)	G14				

Table 7. Adjacency matrix of the gear train graph