# Linear Dual Algebra Algorithms and their Application to Kinematics

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**Summary.** Mathematical and mechanical entities such as line vectors, screws and wrenches can be conveniently represented within the framework of dual algebra. Despite the applications received by this type of algebra, less developed appear the numerical linear algebra algorithms within the field of dual numbers. In this paper will be summarized different basic algorithms for handling vectors and matrices of dual numbers. It will be proposed an original application to finite and infinitesimal rigid body motion analysis.

**Key words:** Clifford algebra, kinematics, screw motion

## 1 Introduction

A *dual number* $\widehat{a}$ is an ordered pair of real numbers $(a, a^o)$ associated with a *real unit* $+1$, and the *dual unit*, or operator $\varepsilon$, where $\varepsilon^2 = \varepsilon^3 = \ldots = 0$, $0\varepsilon = \varepsilon 0 = 0$, $1\varepsilon = \varepsilon 1 = \varepsilon$ . A dual number is usually denoted in the form

$$\widehat{a} = a + \varepsilon a^o \ . \tag{1}$$

A *pure dual number* has the dual unit only.

The algebra of dual numbers has been originally conceived by W.K. Clifford (1873) [9], but its first applications to mechanics are due to A.P. Kotelnikov (1895)[1] and E. Study (1901) [26].

Dual vector algebra provides a convenient tool for handling mathematical entities such as screws and wrenches. In fact, helicoidal infinitesimal and finite rigid body displacements can be easily composed under the framework of dual vector algebra. Another distinctive feature of dual algebra is conciseness of notation. For these reasons it has been often used for the search of closed form solutions in the field of displacement analysis, kinematic synthesis and dynamic analysis of spatial mechanisms.

Dual numbers and their algebra proved to be a powerful tool for the analysis of mechanical systems. Textbooks/monographies entirely dedicated to engineering applications of dual numbers have been authored to F.M Dimentberg [10], R. Beyer [5] and I.S. Fischer [13]. Books with chapters or sections on dual algebra and its applications have been authored by L. Brand [6], M.A. Yaglom [29], J.S. Beggs [4], J. Duffy [11], Gonzáles-Palacios and J. Angeles [15], J. McCarthy [20]. A more extensive list of references is given by E. Pennestrì and R. Stefanelli [22].

One of the purposes of this investigation is the development and implementation of algorithms for the solution of linear algebra problems using dual numbers. Although the algorithms discussed are mainly related to the solution of kinematic problems, we believe they are potentially useful also for dynamic analyses of mechanisms.

The linear algebra algorithms herein presented can be splitted into the following categories:

[1] The original paper of A.P. Kotelnikov, published in the Annals of Imperial University of Kazan (1895), is reputed to have been destroyed during the Russian revolution. [24]

1. simple operations involving dual vectors;
2. dual version of basic algorithms of linear algebra;

whereas the kinematic applications discussed are typical problem in the field of robotics and biomechanics:

1. Given the initial and final positions of the a given number of points on a body, compute the screw motion parameters of such body.
2. Given the velocities of a set of points of a body, compute the instantaneous screw motion parameters.

Regarding the first problem, the solution herein proposed appears original and computationally more attractive than the one reported in [27].

Our algorithm for computing the screw axis of an infinitesimal motion, given the velocities of a set of points, is entirely developed within the field of dual numbers and alternative to the one proposed by Page *et al.* [21]. In this last reference, the equations are splitted into real and dual parts before their numerical solution.

The kinematic algorithms, implemented using the `Ch` programming language[2], can be applied also when numerical data are affected by measurement errors and rigid body properties are not strictly satisfied. In this case, the screw motion parameters of a *pseudo* rigid body motion are computed through the application of the least squares criterion.

## 2 Dual numbers

Dual numbers can be represented as follows:

- Gaussian representation: $\widehat{a} \equiv a + \varepsilon a^o$.
- Polar representation: $\widehat{a} \equiv \rho\,(1 + \varepsilon t)$, where $\rho = a$ and $t = a^o/a$.
- Exponential representation: $\widehat{a} \equiv \rho \mathrm{e}^{\varepsilon t}$, where $\rho = a$, $t = a^o/a$ and $\mathrm{e}^{\varepsilon t} = 1 + \varepsilon t$.

The adoption of one representation instead of another depends on the context.

## 3 Dual functions

A function $F$ of a dual variable $\widehat{x} = x + \varepsilon x^o$ can be represented in the form

$$F\left(\widehat{x}\right) = f\left(x, x^o\right) + \varepsilon g\left(x, x^o\right) \ ,$$

where $f$ and $g$ are real functions of real variables $x$ and $x^o$. The necessary and sufficient conditions in order $F$ be analytic are [10]

$$\frac{\partial f}{\partial x^o} = 0 \ , \qquad \frac{\partial f}{\partial x} = \frac{\partial g}{\partial x^o} \ . \tag{2}$$

From these follows

$$f\left(\widehat{x}\right) = f\left(x + \varepsilon x^o\right) = f(x) + x^o \frac{\partial f}{\partial x} \ . \tag{3}$$

[2] The `Ch` programming language has been developed by Prof. H.H. Cheng of University of California at Davis. More info about its main features are available at the web page `www.softintegration.com`.

## 4 Algebra of dual vectors

A *line vector* is a vector bound to a definite line $\mathcal{L}$ in space. The *dual vector*

$$\widehat{V} = \mathbf{v} + \varepsilon \mathbf{v}^o \tag{4}$$

is combination of two vectors which specifies the position of $\mathcal{L}$ with respect to an arbitrary origin $O$. The *primary part* is a vector $\mathbf{v}$ parallel to $\mathcal{L}$ and the *dual part* is $\mathbf{v}^o = \overrightarrow{OP} \times \mathbf{v}$, where $P$ is an arbitrary point on $\mathcal{L}$.

### 4.1 Scalar and cross products of dual vectors

With reference to Figure 1,

$$\widehat{A} = \mathbf{a} + \varepsilon \left(\mathbf{r}_1 \times \mathbf{a}\right) \ ,$$
$$\widehat{B} = \mathbf{b} + \varepsilon \left(\mathbf{r}_2 \times \mathbf{b}\right) \ ,$$

be two *dual vectors* representing two distinct line vectors and let $\mathbf{s}^*$ the direction versor of the minimum distance between these line vectors directed from $\mathbf{a}$ to $\mathbf{b}$.

In such a context, it is necessary to introduce the concept of *dual angle* [26]

$$\widehat{\theta} = \theta + \varepsilon s \tag{5}$$

as a variable required to characterize the relative position and orientation of line vectors $\widehat{A}$ and $\widehat{B}$. The angle $\theta$ is measured counterclockwise about $\mathbf{s}^*$.

**Table 1.** Computational cost of operations with dual numbers

| **Dual operation** | **Mathematical expression** | Mult. and Div. | Sums | Trig. evaluations |
|---|---|---|---|---|
| Sum | $\widehat{a} \pm \widehat{b} = (a \pm b) + \varepsilon (a^o \pm b^o)$ | - | 2 | - |
| Product | $\widehat{a}\widehat{b} = ab + \varepsilon (a^o b + ab^o)$ | 3 | 1 | - |
| Division[1] | $\dfrac{\widehat{a}}{\widehat{b}} = \dfrac{a}{b} + \varepsilon \dfrac{a^o b - ab^o}{b^2}$ | 5 | 1 | - |
| Square root | $\sqrt{\widehat{a}} = \sqrt{a} + \varepsilon \dfrac{a^o}{2\sqrt{a}}$ | 2 | - | - |
| Vector scaling | $\widehat{a}\,\{\widehat{n}\}$ | 9 | 3 | - |
| Dot product[2] | $\left\{\widehat{A}\right\}^T \left\{\widehat{B}\right\} = \{A\}^T \{B\} + \varepsilon \left(\{A^o\}^T \{B\} + \{B^o\}^T \{A\}\right)$ | 9 | 7 | - |
| Cross product | $\left[\widetilde{\widehat{A}}\right] \left\{\widehat{B}\right\} = \left[\widetilde{A}\right] \{B\} + \varepsilon \left(\{A\}^T \{B^o\} + \{A^o\}^T \{B\}\right)$ | 18 | 12 | - |
| Dual sin | $\sin\widehat{\theta} = \sin\theta + \varepsilon d \cos\theta$ | 1 | 0 | 2 |
| Dual cos | $\cos\widehat{\theta} = \cos\theta - \varepsilon d \sin\theta$ | 1 | 0 | 2 |
| Dual tan | $\tan\widehat{\theta} = \tan\theta + \varepsilon \dfrac{d}{\cos^2\theta}$ | 2 | 0 | 2 |
| Dual asin | $\arcsin\widehat{a} = \arcsin a + \varepsilon \dfrac{a^o}{\sqrt{1-a^2}}$ | 3[3] | 1 | 1 |
| Dual acos | $\arccos\widehat{a} = \arccos a - \varepsilon \dfrac{a^o}{\sqrt{1-a^2}}$ | 3[3] | 1 | 1 |
| Dual atan | $\arctan\widehat{a} = \arctan a + \varepsilon \dfrac{a^o}{1+a^2}$ | 2 | 1 | 1 |

[1] Division by a pure dual number $\varepsilon b^o$ is not defined.
[2] We assume vectors of 3 elements.
[3] The computational cost of a square root operation on real numbers not included in this table is usually considered equivalent to about 8 multiplications/divisions.
[4] $\sim$ denotes a skew-symmetric matrix.

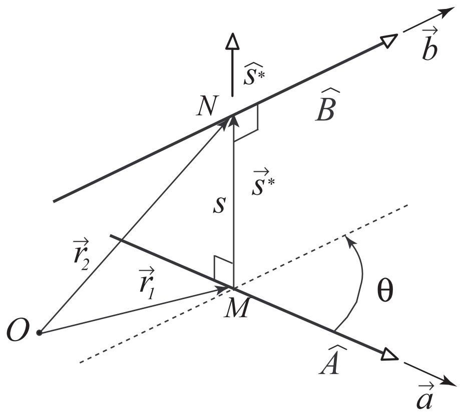


**Fig. 1.** Product of dual vectors:Nomenclature

The scalar and cross products of two dual vectors are respectively defined as follows [10]

$$\begin{aligned} \widehat{A} \cdot \widehat{B} &= \mathbf{a} \cdot \mathbf{b} + \varepsilon \left[ \mathbf{a} \cdot (\mathbf{r}_2 \times \mathbf{b}) + \mathbf{b} \cdot (\mathbf{r}_1 \times \mathbf{a}) \right] \\ &= \mathbf{a} \cdot \mathbf{b} + \varepsilon \left[ (\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{a} \times \mathbf{b}) \right] \\ &= ab \cos\theta - \varepsilon \left[ (s\mathbf{s}^\star) \cdot (ab \sin\theta \mathbf{s}^\star) \right] \\ &= ab \left[ \cos\theta - \varepsilon s \sin\theta \right] = ab \cos\widehat{\theta} \ . \end{aligned} \tag{6}$$

$$\begin{aligned} \widehat{A} \times \widehat{B} &= \mathbf{a} \times \mathbf{b} + \varepsilon \left[ \mathbf{a} \times (\mathbf{r}_2 \times \mathbf{b}) + (\mathbf{r}_1 \times \mathbf{a}) \times \mathbf{b} \right] \\ &= \mathbf{a} \times \mathbf{b} + \varepsilon \left[ (\mathbf{a} \cdot \mathbf{b}) (\mathbf{r}_2 - \mathbf{r}_1) + \mathbf{r}_1 \times (\mathbf{a} \times \mathbf{b}) \right] \\ &= ab \left\{ \mathbf{s}^\star \sin\theta + \varepsilon \left[ s \cos\theta \mathbf{s}^\star + \sin\theta \left( \mathbf{r}_1 \times \mathbf{s}^\star \right) \right] \right\} \\ &= ab\widehat{S}^\star \left( \sin\theta + \varepsilon s \cos\theta \right) = ab\widehat{S}^\star \sin\widehat{\theta} \ , \end{aligned} \tag{7}$$

Table 2 summarizes the result of different dual vectors products for different cases of relative position of line vectors.

**Table 2.** Noteworthy cases of dual vectors products (Adapted from [10])

| Line vector | $\widehat{A} \cdot \widehat{B}$ | $\widehat{A} \times \widehat{B}$ |
|---|---|---|
| Skew | $ab \cos\widehat{\theta}$ | $ab\widehat{S}^\star \sin\widehat{\theta}$ |
| Incident ($s = 0$) | $ab \cos\theta$ | $ab\widehat{S}^\star$ |
| Parallel ($\theta = 0$) | $ab$ | $\varepsilon ab\mathbf{s}^\star$ |
| Coaxial ($\theta = s = 0$) | $ab$ | 0 |

## 4.2 Dual angle between line vectors

The dual angle between the two line vectors

$$\widehat{A}_i = \mathbf{a}_i + \varepsilon \left( \mathbf{s}_i \times \mathbf{a}_i \right) \ . \quad (i = 1, 2)$$

must be computed. The computational steps are described in the following and are justified by the geometry depicted in Figure 2.

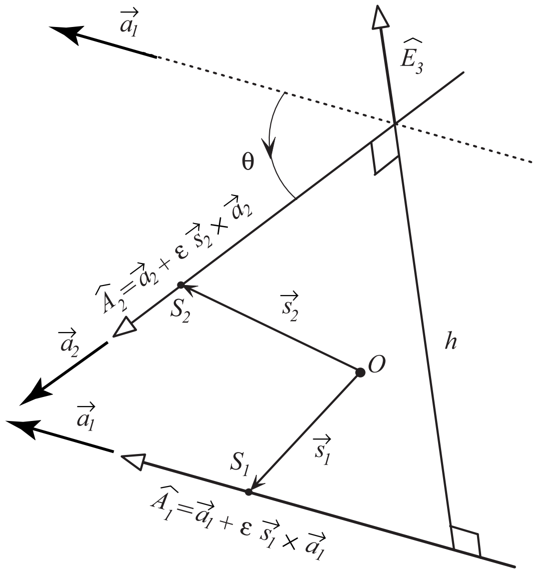


**Fig. 2.** Computation of dual angle between two line vectors

1. Compute the dual vectors
$$\widehat{E}_i = \frac{\widehat{A}_i}{\left\|\widehat{A}_i\right\|} \quad (i = 1, 2) \tag{8}$$

2. Compute their cross product
$$\widehat{E}_3 = \frac{\widehat{E}_1 \times \widehat{E}_2}{\left\|\widehat{E}_1 \times \widehat{E}_2\right\|} \ . \tag{9}$$

3. Compute cosine and sine of the dual angle $\widehat{\theta}$ between the two line vectors
$$\cos\widehat{\theta} = \widehat{E}_1 \cdot \widehat{E}_2 \ , \tag{10a}$$
$$\sin\widehat{\theta} = \widehat{E}_1 \times \widehat{E}_2 \cdot \widehat{E}_3 \ . \tag{10b}$$

4. Compute dual angle
$$\widehat{\theta} = \mathrm{atan2}\left(\sin\widehat{\theta}, \cos\widehat{\theta}\right) = \theta + \varepsilon h \ . \tag{11}$$

The procedure is not valid if line vectors are parallel. In this case, there is an infinite set of dual vectors $\widehat{E}_3$.

### 4.3 Sum of two dual vectors

With reference to the geometry of Figure 3, we wish compute the sum
$$\widehat{A} = \widehat{A}_1 + \widehat{A}_2 \tag{12}$$

One can observe that the direction of $\widehat{A}$ is obtained by prescribing a screw motion to $\widehat{A}_1$ defined by the screw axis $\widehat{E}_{12}$ and dual angle $\widehat{\alpha_1}$. On the basis of this observation, the following algorithm can be stated:

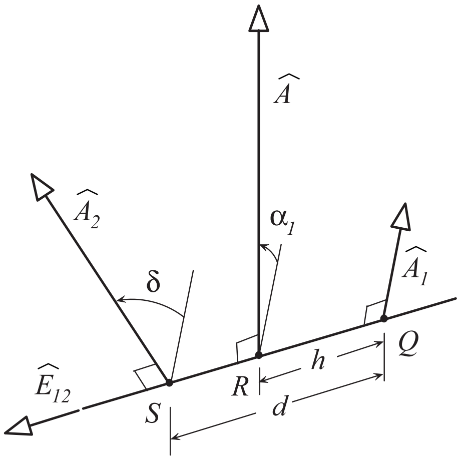


**Fig. 3.** Sum of dual vectors

1. Compute the dual vectors

$$\widehat{E}_1 = \frac{\widehat{A}_1}{\left\|\widehat{A}_1\right\|} ,$$

$$\widehat{E}_2 = \frac{\widehat{A}_2}{\left\|\widehat{A}_2\right\|} ,$$

where $\|\cdot\|$ denote the Euclidean norm.
2. Compute the dual angle $\widehat{\theta}$ and the dual vector $\widehat{E}_{12}$ perpendicular to both $\widehat{A}_1$ and $\widehat{A}_2$ (see previous section).
3. Compute the module of the dual vector sum

$$A = \sqrt{\widehat{A}_1 \cdot \widehat{A}_1 + \widehat{A}_2 \cdot \widehat{A}_2 + 2\widehat{A}_1 \cdot \widehat{A}_2}$$

4. Compute sine and cosine of dual angle $\widehat{\alpha}_1$

$$\sin\widehat{\alpha}_1 = \frac{\sin\widehat{\theta}\sqrt{\widehat{A}_2 \cdot \widehat{A}_2}}{A} ,$$

$$\cos\widehat{\alpha}_1 = \frac{\sqrt{\widehat{A}_1 \cdot \widehat{A}_1}}{A} + \sin\widehat{\alpha}_1 \frac{\cos\widehat{\theta}}{\sin\widehat{\theta}} ,$$

$$\widehat{\alpha}_1 = \operatorname{atan2}\left(\sin\widehat{\alpha}_1, \cos\widehat{\alpha}_1\right)$$

5. Let

$$\widehat{t} = \tan\frac{\widehat{\alpha}_1}{2}$$

and compute

$$\widehat{T} = \widehat{t}\widehat{E}_{12} . \tag{13}$$

6. Apply formula [3]

$$\widehat{E} = \widehat{E}_1 + \frac{2\widehat{T}}{1+\widehat{t}^2} \times \left(\widehat{E}_1 + \widehat{T} \times \widehat{E}_1\right)$$

for the definition of $\widehat{E}_1$ after the screw motion

[3] As will be explained in subsection 7.1, this formula represents the extension to dual algebra of the well known Rodrigues' formula.

7. Finally compute

$$\widehat{A} = A\widehat{E} \ .$$

***Example:*** If we assume

$$\widehat{A}_1 = \{\ 0\ 0\ 1\ \}^T \ , \quad \widehat{A}_2 = \{\ 0\ 1\ \varepsilon\ \}^T \ ,$$

the previous algorithm give the following numerical values

$$\widehat{\theta} = \frac{\pi}{2} - \varepsilon \ ,$$

$$\widehat{E}_{12} = \{\ -1\ 0\ 0\ \}^T \ ,$$

$$\widehat{\alpha}_1 = \frac{\pi}{4} - \frac{\varepsilon}{2} \ ,$$

$$\widehat{A} = \{\ 0\ 1\ 1+\varepsilon\ \}^T \ .$$

The loci formed by all possible $\widehat{A}$ form a ruled surface named *cylindroid*. Ready to use computer-aided solutions of this problem have been also proposed by A.Perez [23].

## 5 Solution of matrix equations

When adapting linear algebra numerical algorithms to matrices of dual numbers, one may encounter the following matrix equation

$$[X]\,[L]^T + [L]\,[X]^T = [A^o] \ , \tag{14}$$

where $[A^o]$ and $[L]$ are prescribed whereas the elements of matrix $[X]$ are unknown.

Equation (14) can be transformed into the following

$$([L] \otimes [I])\,\mathrm{vec}\,([X]) + ([I] \otimes [L])\,\mathrm{vec}\left([X]^T\right) = \mathrm{vec}\,([A^o]) \tag{15}$$

where $\otimes$ and $\mathrm{vec}(\cdot)$ denote the Kronecker product between two matrices and the vectorization of a matrix, respectively.

If we let $[H_1] = [L]\otimes[I]$ and $[H_2]$ the matrix obtained from $([I] \otimes [L])$ exchanging columns appropriately, then (15) can be rewritten in the form

$$[H]\,\mathrm{vec}\,([X]) = \mathrm{vec}\,([A^o]) \ . \tag{16}$$

where $[H] = [H_1] + [H_2]$. Since $[H]$ is a singular matrix, further conditions need to be added for a unique solution of the problem. For instance, in the case of Cholesky decomposition one must delete the redundant rows of $[H]$ and require that the elements of $[X]$ above the main diagonal are all zero.

## 6 Algebra of dual matrices

A dual matrix $\left[\widehat{A}\right]$ is a matrix whose components are dual numbers. It can be splitted into a real part $[A]$ and a dual part $[A^o]$ such that

$$\left[\widehat{A}\right] = [A] + \varepsilon\,[A^o] \ . \tag{17}$$

### 6.1 Product of two dual matrices

Assuming that $\left[\widehat{A}\right] = [A] + \varepsilon\,[A^o]$ and $\left[\widehat{B}\right] = [B] + \varepsilon\,[B^o]$ have the appropriate dimensions, then their dual product is defined as follows:

$$\left[\widehat{A}\right]\left[\widehat{B}\right] = [A]\,[B] + \varepsilon\,([A]\,[B^o] + [A^o]\,[B]) \tag{18}$$

### 6.2 Inverse of a dual matrix

The inverse of a square dual matrix is defined as

$$\left[\widehat{A}\right]\left[\widehat{A}\right]^{-1} = [I] \ . \tag{19}$$

By letting $\left[\widehat{B}\right] = \left[\widehat{A}\right]^{-1}$, from (18) one obtains [16, 3]

$$\left[\widehat{A}\right]^{-1} = [A]^{-1} - \varepsilon\left([A]^{-1}[A^o][A]^{-1}\right) \ . \tag{20}$$

### 6.3 QR decomposition of a dual matrix

A dual matrix $\left[\widehat{A}\right]$ can be decomposed as follows

$$\left[\widehat{A}\right] = \left[\widehat{Q}\right]\left[\widehat{R}\right] \tag{21}$$

where $\left[\widehat{Q}\right] = [Q] + \varepsilon [Q^o]$ is an orthogonal matrix and $\left[\widehat{R}\right] = [R] + \varepsilon [R^o]$ is an upper triangular matrix.
For the QR decomposition of matrix $\left[\widehat{A}\right] = [A] + \varepsilon [A^o]$ the authors could not find a formula for obtaining directly the matrices $\left[\widehat{Q}\right]$ and $\left[\widehat{R}\right]$ as a function of $[A]$ and $[A^o]$. Thus, for the computation of $\left[\widehat{Q}\right]$ and $\left[\widehat{R}\right]$, the modified Gram-Schmidt orthogonalization procedure has been applied to the rows of $\left[\widehat{A}\right]$.
To obtain the QR decomposition one can adapt to dual numbers the modified Gram-Schmidt orthogonalization procedure [14].
***Example:*** The matrix

$$\left[\widehat{A}\right] = \begin{bmatrix} 1+\varepsilon & 2+\varepsilon 3 \\ 3+\varepsilon 9 & 3+\varepsilon \end{bmatrix}$$

can be decomposed in the following matrices[4]:

$$\left[\widehat{Q}\right] = \begin{bmatrix} 0.316-\varepsilon 0.569 & 0.949+\varepsilon 0.190 \\ 0.949+\varepsilon 0.190 & -0.316+\varepsilon 0.569 \end{bmatrix} \ , \quad \left[\widehat{R}\right] = \begin{bmatrix} 3.162+\varepsilon 8.854 & 3.478+\varepsilon 1.328 \\ 0.000+\varepsilon 0.000 & 0.948+\varepsilon 4.617 \end{bmatrix} \ .$$

### 6.4 Cholesky decomposition

A symmetric dual matrix $\left[\widehat{A}\right]$ can be decomposed as

$$\left[\widehat{A}\right] = \left[\widehat{L}\right]\left[\widehat{L}\right]^T \tag{22}$$

where $\left[\widehat{L}\right] = [L] + \varepsilon [L^o]$ is a lower triangular matrix with not pure dual numbers diagonal entries linear combination of two lower triangular matrices $[L]$ and $[L^o]$.
***Example:*** The matrix

$$\left[\widehat{A}\right] = \begin{bmatrix} 2+\varepsilon & 1+\varepsilon 4 \\ 1+\varepsilon 4 & 3+\varepsilon \end{bmatrix}$$

can be decomposed as follows

$$[A] + \varepsilon [A^o] = [L][L]^T + \varepsilon\left([L^o][L]^T + [L][L^o]^T\right) \ . \tag{23}$$

The matrix

$$[L] = \begin{bmatrix} 1.414 & 0.00 \\ 0.710 & 1.58 \end{bmatrix}$$

[4] Numerical results are displayed with three decimal digits only.

is immediately computed, whereas the matrix $[L^o]$ is obtained by solving an equation similar to (15), with $[X] = [L^o]$. In this case

$$[H_1] = \begin{bmatrix} 1.414 & 0.00 & 0.00 & 0.00 \\ 0.71 & 1.58 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.414 & 0.00 \\ 0.00 & 0.00 & 0.71 & 1.58 \end{bmatrix} , \quad [H_2] = \begin{bmatrix} 1.414 & 0.00 & 0.00 & 0.00 \\ 0.71 & 0.00 & 1.58 & 0.00 \\ 0.00 & 1.414 & 0.00 & 0.00 \\ 0.00 & 0.71 & 0.00 & 1.58 \end{bmatrix} .$$

The system to be solved is

$$\begin{bmatrix} 2.828 & 0.00 & 0.00 & 0.00 \\ 0.71 & 1.414 & 1.58 & 0.00 \\ 0.00 & 1.42 & 0 & 3.16 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 4 \\ 1 \\ 0 \end{Bmatrix} ,$$

hence

$$[L^o] = \begin{bmatrix} 0.35 & 0 \\ 2.65 & -0.87 \end{bmatrix} .$$

### 6.5 Pseudoinverse of a dual matrix

Let $\left[\widehat{A}\right]$ be a matrix with $m$ rows and $n$ columns. Its dual Moore-Penrose pseudoinverse is defined as follows:

$$\left[\widehat{A}\right]^+ = \begin{cases} \left(\left[\widehat{A}\right]^T \left[\widehat{A}\right]\right)^{-1} \left[\widehat{A}\right]^T & \text{if } m \geq n \\ \left[\widehat{A}\right]^T \left(\left[\widehat{A}\right] \left[\widehat{A}\right]^T\right)^{-1} & \text{if } m < n \end{cases} \tag{24}$$

If $m > n$, then the left pseudoinverse exists such that $\left[\widehat{A}\right]^+ \left[\widehat{A}\right] = [I]$. If $m < n$ then the right pseudoinverse exists such that $\left[\widehat{A}\right] \left[\widehat{A}\right]^+ = [I]$. Adopting a reasoning similar to the one used for the definition of the inverse matrix, one can demonstrate that

$$\left[\widehat{A}\right]^+ = [A]^+ - \varepsilon \left([A]^+ [A^o] [A]^+\right) \tag{25}$$

The definition of the Moore-Penrose matrix is often associated with the least squares solution of a linear system, such as (26), but with the number $m$ of equations different from the number $n$ of unknowns. It is well known that $[A]^T [A]$ (or $[A] [A]^T$ ) may not have an inverse and, even when it is invertible, one usually obtains large numerical errors from using directly (24). Thus, to obtain significant numerical answers one must use more sophisticated techniques.

***Example:*** The Moore-Penrose pseudoinverses of

$$\left[\widehat{A}_1\right] = \begin{bmatrix} 1+\varepsilon 4 & 3+\varepsilon 0 \\ 9+\varepsilon 2 & 22+\varepsilon 4 \\ 4+\varepsilon 4 & 4+\varepsilon 1 \end{bmatrix} , \quad \left[\widehat{A}_2\right] = \begin{bmatrix} 1+\varepsilon 4 & 3+\varepsilon 0 & 4+\varepsilon \\ 9+\varepsilon 2 & 22+\varepsilon 4 & 4+\varepsilon 4 \end{bmatrix} .$$

are, respectively,

$$\left[\widehat{A}_1\right]^+ = \begin{bmatrix} -0.051+\varepsilon 0.064 & -0.069+\varepsilon 0.082 & 0.418-\varepsilon 0.533 \\ 0.028-\varepsilon 0.025 & 0.073-\varepsilon 0.038 & -0.170+\varepsilon 0.199 \end{bmatrix} , \quad \left[\widehat{A}_2\right]^+ = \begin{bmatrix} -0.035\varepsilon - 0.014 & 0.021+\varepsilon 0.000 \\ -0.038-\varepsilon 0.035 & 0.044-\varepsilon 0.001 \\ 0.287-\varepsilon 0.007 & -0.038-\varepsilon 0.011 \end{bmatrix} .$$

### 6.6 Solution of a system of dual linear equations

Let us denote with

$$\left[\widehat{A}\right] \{\widehat{x}\} = \left\{\widehat{b}\right\} \tag{26}$$

a system of linear dual equations [8] where $\left[\widehat{A}\right] = [A] + \varepsilon [A^o]$ and $\left\{\widehat{b}\right\} = \{b\} + \varepsilon \{b^o\}$. Assuming $[A]$ nonsingular. The solution $\{\widehat{x}\} = \{x\} + \varepsilon \{x^o\}$ is computed by solving the systems

$$[A] \{x\} = \{b\} , \tag{27a}$$

$$[A] \{x^o\} = \{b^o\} - [A^o] \{x\} . \tag{27b}$$

To improve the overall computational efficiency it is convenient to factor matrix $[A]$ only once. Thus, we suggest to proceed as follows:

I) Apply QR decomposition to matrix $[A]$ so that $[A] = [Q]\,[R]$.
II) Solve system $[R]\,\{x\} = [Q]^T\,\{b\}$.
III) Solve system $[R]\,\{x^o\} = [Q]^T\,(\{b^o\} - [A^o]\,\{x\})$.
IV) Form the dual vector $\{\widehat{x}\} = \{x\} + \varepsilon\,\{x^o\}$

Since $[R]$ is an upper triangular matrix, at steps II and III only simple procedures of back substitution are executed.

# 7 The Principle of Transference

The Principle of Transference proved to be a powerful tool for kinematic analysis of spatial linkages. The Principle has been declared by A.P Kotelnikov for the first time, but the correspondence between equivalent spherical and spatial configurations is due to V.V. Dobrovolski (1947).

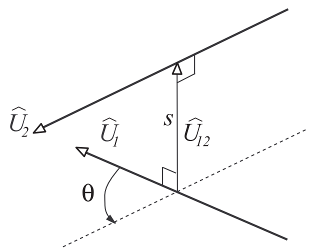


**Fig. 4.** Nomenclature

A thoughtful discussion of the Principle of Transference is given by J. Rooney [24], L.M. Hsia and A.T. Yang [18].

The Principle of Transference can be stated as follows [10, 24, 18]:

> All valid laws and formulae relating to a system of intersecting unit line vectors (and hence involving real variables) are equally valid when applied to an equivalent system of skew vectors, if each variable $a$, in the original formulae is replaced by the corresponding dual variable $\widehat{a} = a + \varepsilon a^o$.

## 7.1 Applications of the Principle of Transference

By virtue of the Principle of Transference, formulas for the composition of spherical motions can be extended to the general helicoidal motion case by simply substituting the angle of rotation $\theta$ with the *dual angle* $\widehat{\theta} = \theta + \varepsilon s$, where $s$ is the displacement of the body along the screw axis.

### Extension of the Rodrigues' formula

With reference to the geometry of Figure 5, let $\mathbf{r}_2$ be the position of vector $\mathbf{r}_1$ after a rotation about axis of versor $\mathbf{u}$ of an angle $\theta$.

The well known Rodrigues' formula, in vector notation, can be rewritten in the form

$$\mathbf{r}_2 = \mathbf{r}_1 + \frac{2\boldsymbol{\tau}}{1+t^2} \times (\mathbf{r}_1 + \boldsymbol{\tau} \times \mathbf{r}_1) \ , \qquad (28)$$

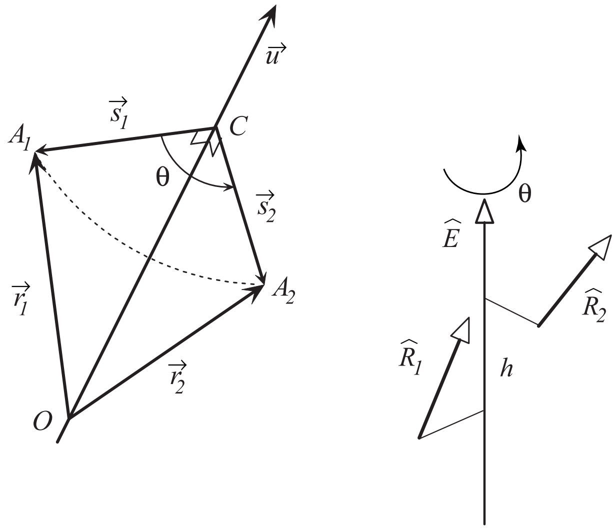


**Fig. 5.** Spherical motion (left) and screw motion about axis $h$ (right)

where $t = \tan\frac{\theta}{2}$ and $\boldsymbol{\tau} = t\mathbf{u}$. When $\theta = \pi$ the previous expression cannot be applied and the following should be adopted

$$\mathbf{r}_2 = 2\,(\mathbf{u}\cdot\mathbf{r}_1)\,\mathbf{u} - \mathbf{r}_1\ . \tag{29}$$

By applying the principle of Transference, the Rodrigues' formula (28) can be generalised to define a screw motion about the line vector defined by $\widehat{E}$ as follows

$$\widehat{R}_2 = \widehat{R}_1 + \frac{2\widehat{T}}{1+\tan^2\frac{\widehat{\theta}}{2}} \times \left(\widehat{R}_1 + \widehat{E}\tan\frac{\widehat{\theta}}{2}\times\widehat{R}_1\right)\ , \tag{30}$$

where

- $\widehat{R}_1$ and $\widehat{R}_2$ are the initial and final positions of a line vector framed to the rigid body, respectively (see Figure 5);
- $\widehat{\theta} = \theta + \varepsilon s$ is the dual angle whose primary part is the angle of rotation and $s$ the displacement along the screw axis.

### Composition of finite screw motions

Given two spherical rotations, the first one of an angle $\theta_1$ about the axis $\mathbf{u}_1$, and the second of an angle $\theta_2$ abuot the axis $\mathbf{u}_2$. Let us introduce the vector

$$\boldsymbol{\tau}_i = \tan\frac{\theta_i}{2}\mathbf{u}_i \quad (i = 1, 2)\ , \tag{31}$$

It can be demonstrated [19, 20] that the resultant spherical motion is defined by the following vector

$$\boldsymbol{\tau}_3 = \frac{\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 - \boldsymbol{\tau}_1\times\boldsymbol{\tau}_2}{1 - \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}\ . \tag{32}$$

Consider two finite screw motions about the axes located by the unit line dual vector $\widehat{E}_i$ $(i = 1, 2)$. The rotation angle and the displacement along the axis are denoted with $\theta_i$ and $h_i$ $(i = 1, 2)$, respectively.

The formula for the composition of these finite motions is obtained by changing the vectors into dual vectors

$$\widehat{T}_3 = \frac{\widehat{T}_1 + \widehat{T}_2 - \widehat{T}_1 \times \widehat{T}_2}{1 - \widehat{T}_1 \cdot \widehat{T}_2} \ , \tag{33}$$

where

$$\widehat{T}_i = \widehat{E}_i \tan \frac{\widehat{\theta}_i}{2} \ , (i = 1, 2, \ldots) \tag{34}$$

### The dual screw matrix

Let $\widehat{E} = \left\{ \widehat{h}_1 \; \widehat{h}_2 \; \widehat{h}_3 \right\}^T$ and $\widehat{\theta}$ be the line versor representing the screw axis of the finite rigid body motion and the dual rotation angle $\widehat{\theta}$, then, the orthogonal transform matrix representing this motion can be written in the form

$$\left[\widehat{A}\right] = \begin{bmatrix} \widehat{a}_{11} & \widehat{a}_{12} & \widehat{a}_{13} \\ \widehat{a}_{21} & \widehat{a}_{22} & \widehat{a}_{23} \\ \widehat{a}_{31} & \widehat{a}_{32} & \widehat{a}_{33} \end{bmatrix} = 2 \begin{bmatrix} \widehat{e}_0^2 + \widehat{e}_1^2 - \frac{1}{2} & \widehat{e}_1\widehat{e}_2 - \widehat{e}_0\widehat{e}_3 & \widehat{e}_1\widehat{e}_3 + \widehat{e}_0\widehat{e}_2 \\ \widehat{e}_1\widehat{e}_2 + \widehat{e}_0\widehat{e}_3 & \widehat{e}_0^2 + \widehat{e}_2^2 - \frac{1}{2} & \widehat{e}_2\widehat{e}_3 - \widehat{e}_0\widehat{e}_1 \\ \widehat{e}_1\widehat{e}_3 - \widehat{e}_0\widehat{e}_2 & \widehat{e}_2\widehat{e}_3 + \widehat{e}_0\widehat{e}_1 & \widehat{e}_0^2 + \widehat{e}_3^2 - \frac{1}{2} \end{bmatrix} . \tag{35}$$

where

$$\widehat{a}_{ij} = a_{ij} + \varepsilon a_{ij}^o \quad (i, j = 1, 2, 3)$$

and

$$\widehat{e}_0 = \cos \frac{\widehat{\theta}}{2} \ , \quad \widehat{e}_1 = \widehat{h}_1 \sin \frac{\widehat{\theta}}{2} \ , \quad \widehat{e}_2 = \widehat{h}_2 \sin \frac{\widehat{\theta}}{2} \ , \quad \widehat{e}_3 = \widehat{h}_3 \sin \frac{\widehat{\theta}}{2} \ . \tag{36}$$

are the dual Euler parameters.

The matrix (35) transforms the dual coordinates of a line vector attached to a rigid body undergoing a screw motion from its initial to the final position.

### Computing screw motion parameters from the dual screw matrix

In this section it will be discussed how to obtain screw motion parameters such as position and orientation of the screw axis versor $\widehat{E}$ and dual angle $\widehat{\theta} = \theta + \varepsilon s$ about such axis. The problem is solved by different authors (*e.g.* [1]). However,the approaches of I.S. Fischer [12] and H.H. Cheng [7] rely upon dual numbers. However, in the algorithm herein proposed all computations are entirely made in the field of dual numbers, *i.e.* no splitting of matrix elements into a real and dual part is required.

1. Compute

$$\widehat{e}_0 = \pm \frac{\sqrt{1 + \widehat{a}_{11} + \widehat{a}_{22} + \widehat{a}_{33}}}{2} \ , \quad \widehat{e}_1 = \pm \frac{\sqrt{1 + \widehat{a}_{11} - \widehat{a}_{22} - \widehat{a}_{33}}}{2} \ ,$$
$$\widehat{e}_2 = \pm \frac{\sqrt{1 - \widehat{a}_{11} + \widehat{a}_{22} - \widehat{a}_{33}}}{2} \ , \quad \widehat{e}_3 = \pm \frac{\sqrt{1 - \widehat{a}_{11} - \widehat{a}_{22} + \widehat{a}_{33}}}{2} \ .$$

   The choice of the algebraic sign affects only the sense of the versor of the screw axis.

2. Avoiding the division by pure dual numbers, choose the appropriate $k^{\text{th}}$ case for the computation of the remaining dual Euler parameters.

| Case 0 | Case 1 | Case 2 | Case 3 |
|---|---|---|---|
| $\widehat{e}_1 = \dfrac{\widehat{a}_{32} - \widehat{a}_{23}}{4\widehat{e}_0}$ | $\widehat{e}_2 = \dfrac{\widehat{a}_{12} + \widehat{a}_{21}}{4\widehat{e}_1}$ | $\widehat{e}_3 = \dfrac{\widehat{a}_{23} + \widehat{a}_{32}}{4\widehat{e}_2}$ | $\widehat{e}_2 = \dfrac{\widehat{a}_{32} + \widehat{a}_{23}}{4\widehat{e}_3}$ |
| $\widehat{e}_2 = \dfrac{\widehat{a}_{13} - \widehat{a}_{31}}{4\widehat{e}_0}$ | $\widehat{e}_3 = \dfrac{\widehat{a}_{13} + \widehat{a}_{31}}{4\widehat{e}_1}$ | $\widehat{e}_0 = \dfrac{\widehat{a}_{13} - \widehat{a}_{31}}{4\widehat{e}_2}$ | $\widehat{e}_1 = \dfrac{\widehat{a}_{13} + \widehat{a}_{31}}{4\widehat{e}_3}$ |
| $\widehat{e}_3 = \dfrac{\widehat{a}_{21} - \widehat{a}_{12}}{4\widehat{e}_0}$ | $\widehat{e}_0 = \dfrac{\widehat{a}_{32} - \widehat{a}_{23}}{4\widehat{e}_1}$ | $\widehat{e}_1 = \dfrac{\widehat{a}_{12} + \widehat{a}_{21}}{4\widehat{e}_2}$ | $\widehat{e}_0 = \dfrac{\widehat{a}_{21} - \widehat{a}_{12}}{4\widehat{e}_3}$ |

3. Compute

$$\widehat{\theta} = 2\cos^{-1}\widehat{e}_k \tag{37a}$$

$$\widehat{h}_i = \frac{\widehat{e}_i}{\sin\frac{\widehat{\theta}}{2}} \quad (i = 1, 2, 3) \tag{37b}$$

Our procedure is still valid for half-turn screw motion, whereas the one proposed by I.S. Fischer [12] needs to be modified. For the case of pure translation (*i.e* $\theta = 0°$) (37b) cannot be applied. It must be observed that under a pure translation there is a unique motion direction, but not a unique screw axis. Hence, the dual part of the screw versor is meaningless and need not be calculated. If a pure translation occurs, then the direction of the motion axis is given by [12]

$$h_1 = \frac{a_{32}}{s}\ , \quad h_2 = \frac{a_{13}}{s}\ , \quad h_3 = \frac{a_{21}}{s}\ . \tag{38}$$

## 8 Computing screw finite motion parameters from redundant noisy landmark data

In the field of biomechanics or robotics it is often required the computation of body screw motion parameters from the Cartesian coordinates of tracked markers attached to the body [2, 28, 25, 17].

Since these coordinates are affected by experimental errors of different nature, the condition of constant distance between the centers of the markers is not satisfied. Therefore, the tracked object cannot be strictly considered a rigid body. It is well known that the trajectories of three non aligned points are required to uniquely specify a rigid body motion. With the purpose to reduce the influence of experimental errors, the number of tracked points is usually more than three. Such redundancy, although beneficial for the accuracy of computations, even in the case of data affected by errors, rules out algorithms based on the hypothesis of rigid body motion (*e.g.* [19, 1]).

The technical literature reports several techniques which reduce the motion of the tracked object to a pseudo rigid body motion. Error smoothing is usually based on the least squares criterion. In biomechanics, the use of dual numbers for the solution of the problem is not new [27].

Let us denote with $\{r_{0i}\}$ and $\{r_i\}$ $(i = 1, 2, \ldots, n)$, respectively, the initial and final coordinates of points attached to a body subjected to a screw motion. These coordinates are collected through different experimental techniques such as photogrammetry, magnetic sensors, laser sensors, etc. The initial and final positions of the centroid of the points are given by

$$\{c_{0i}\} = \frac{1}{n}\sum_i \{r_{0i}\} \tag{39a}$$

$$\{c_i\} = \frac{1}{n}\sum_i \{r_i\}\ . \tag{39b}$$

The initial and final positions of line vectors attached to the moving body are expressed, respectively, by the following dual vectors

$$\{\widehat{r}_{0i}\} = \{r_{0i} - c_{0i}\} + \varepsilon\,[\widetilde{c}_{0i}]\,\{r_{0i} - c_{0i}\} \ , \tag{40a}$$

$$\{\widehat{r}_{i}\} = \{r_{i} - c_{i}\} + \varepsilon\,[\widetilde{c}_{i}]\,\{r_{i} - c_{i}\} \ , \tag{40b}$$

where the symbol $\widetilde{\cdot}$ denotes the skew-symmetric matrix of a vector.

In the absence of errors, the following equality would hold:

$$\left[\widehat{A}\right]\{\widehat{r}_{0i}\} = \{\widehat{r}_i\} \ , \tag{41}$$

with $\left[\widehat{A}\right]$ expressed by (35). However, due to the presence of errors

$$\left[\widehat{A}_1\right]\{\widehat{r}_{0i}\} \approx \{\widehat{r}_i\} \ , \tag{42}$$

where $\left[\widehat{A}_1\right]$ is an unknown matrix to be computed trying to minimize the differences with the least squares optimality criterion.

In this section a two step method is proposed:

1. After forming the matrices

$$\left[\widehat{\mathsf{R}}_0\right] = \left[\,\widehat{r}_{01}\ \widehat{r}_{02}\ \ldots\ \widehat{r}_{0n}\,\right] \tag{43a}$$

$$\left[\widehat{\mathsf{R}}_1\right] = \left[\,\widehat{r}_{1}\ \widehat{r}_{2}\ \ldots\ \widehat{r}_{n}\,\right] \tag{43b}$$

   a dual tranform $\left[\widehat{A}_1\right]$ matrix is simply obtained as follows

$$\left[\widehat{A}_1\right] = \left[\widehat{\mathsf{R}}_1\right]\left[\widehat{\mathsf{R}}_0\right]^{+} \tag{44}$$

   Then the dual QR decomposition is applied

$$\left[\widehat{A}_1\right] = \left[\widehat{Q}\right]\left[\widehat{R}\right] \tag{45}$$

   and we let

$$\left[\widehat{A}\right] = \left[\widehat{Q}\right] \ . \tag{46}$$

   Under ideal conditions the matrix $\left[\widehat{R}\right]$ will result into an identity matrix. Hence, its elements can be used as a rough estimate of the deviation from rigid body condition.

2. By means of the algorithm discussed in the previous section, from $[A]$ the screw motion parameters $\widehat{\theta}$ and $\widehat{E}$ are then retrieved.

The present method offers the following advantages:

- it is entirely based on dual equations and this greatly reduces the number of equations and the order of matrices involved;
- no iterative optimization solver is required.

## 9 Computing screw infinitesimal motion parameters

Let $\{p_i\}$, $\{\dot{p}_i\}$ $i = 1, 2, \ldots, n$ be the positions and velocities of a set of points of a rigid body and

$$\{g\} = \frac{1}{n}\sum_i \{p_i\} \tag{47a}$$

$$\{\dot{g}\} = \frac{1}{n}\sum_i \{\dot{p}_i\} \tag{47b}$$

$$\{\widehat{r}_i\} = \{p_i - g\} + \varepsilon[\widetilde{g}]\{p_i - g\} \tag{47c}$$

$$\frac{d}{dt}\{\widehat{r}_i\} = \{\dot{p}_i - \dot{g}\} + \varepsilon[\widetilde{\dot{g}}]\{p_i - g\} + \varepsilon[\widetilde{g}]\{\dot{p}_i - \dot{g}\} \tag{47d}$$

From the rule of differention of vectors, the following equality follows

$$\{\widehat{v}_i\} = \frac{d}{dt}\{\widehat{r}_i\} = \left[\widetilde{\widehat{\omega}}\right]\{\widehat{r}_i\} \tag{48}$$

where $\{\widehat{\omega}\} = \left\{\, \widehat{\omega}_x \; \widehat{\omega}_y \; \widehat{\omega}_z \,\right\}^T$ is the dual angular velocity vector.

If $\{\widehat{u}\}$ is the dual vector which defines the spatial position of the instantaneous screw axis of a rigid body, $\omega$ the angular velocity, $v$ the velocity of the points on such axis, then the dual angular velocity vector is defined as follows

$$\{\widehat{\omega}\} = (\omega + \varepsilon v)\,\{\widehat{u}\}\ . \tag{49}$$

This dual vector characterize the entire field of velocities of all the points on the rigid body.

In some applications, given the velocities of some points on a rigid body, it is necessary to compute $\{\widehat{\omega}\}$. For $n \geq 3$, then from (48) one obtains

$$\begin{Bmatrix} \widehat{v}_{1x} \\ \widehat{v}_{1y} \\ \widehat{v}_{1z} \\ \widehat{v}_{2x} \\ \widehat{v}_{2y} \\ \widehat{v}_{2z} \\ \vdots \\ \widehat{v}_{nx} \\ \widehat{v}_{ny} \\ \widehat{v}_{nz} \end{Bmatrix} = - \begin{bmatrix} 0 & -\widehat{r}_{1z} & \widehat{r}_{1y} \\ \widehat{r}_{1z} & 0 & -\widehat{r}_{1x} \\ -\widehat{r}_{1y} & \widehat{r}_{1x} & 0 \\ 0 & -\widehat{r}_{2z} & \widehat{r}_{2y} \\ \widehat{r}_{2z} & 0 & -\widehat{r}_{2x} \\ -\widehat{r}_{2y} & \widehat{r}_{2x} & 0 \\ & \vdots & \\ & & 0 \\ 0 & -\widehat{r}_{nz} & \widehat{r}_{ny} \\ \widehat{r}_{nz} & 0 & -\widehat{r}_{nx} \\ -\widehat{r}_{ny} & \widehat{r}_{nx} & 0 \end{bmatrix} \begin{Bmatrix} \widehat{\omega}_x \\ \widehat{\omega}_y \\ \widehat{\omega}_z \end{Bmatrix} \tag{50}$$

This can be interpreted as a redundant system of linear equations with the dual unknowns $\widehat{\omega}_x$, $\widehat{\omega}_y$ and $\widehat{\omega}_z$. Its solution can be obtained using the pseudoinverse matrix of the matrix coefficients.

However, in order to obtain meaningful results, one must ensure that all the points whose velocities are prescribed do not lie on a single line or belong to a plane parallel with the infinitesimal screw axis.

The equation (50) can be applied also for the case of velocity data affected by measurement errors. Obviously, the computed $\{\widehat{\omega}\}$ approximates the real one according to the least squares criterion.

## 10 Numerical applications

Three different examples will be discussed in this section. The first one regards the computation of screw parameters of a rigid body motion using the initail and final coordinates of four point. The second example deals also with the computation of screw motion parameters. In this case, the values of the coordinates are perturbated. Under the new conditions, the hypothesis of rigidity is not valid anymore.

In the third example, the results obtained by solving equation (50) are shown.

**I. Screw parameters for a rigid body finite motion**

The prescribed initial and final positions of four points attached to the body are:

$$\{r_{01}\} = \left\{ 1\ 1\ 1 \right\}^T \ , \quad \{r_{02}\} = \left\{ 1\ 2\ 1 \right\}^T \ ,$$
$$\{r_{03}\} = \left\{ 0\ 2\ 3 \right\}^T \ , \quad \{r_{04}\} = \left\{ 3\ 6\ 7 \right\}^T \ .$$

$$\{r_1\} = \left\{ 2.612370\ 0.387620\ 1.500000 \right\}^T \ , \quad \{r_2\} = \left\{ 2.862370\ 1.137620\ 2.113720 \right\}^T \ ,$$
$$\{r_3\} = \left\{ 3.337110\ -0.337110\ 3.724740 \right\}^T \ , \quad \{r_4\} = \left\{ 9.036607\ 0.963393\ 6.337117 \right\}^T \ .$$

Once the matrices (43) are formed, from (44) follows

$$\left[\widehat{A}\right] = \left[\widehat{A}_1\right] = \begin{bmatrix} 0.750 - \varepsilon 0.250 & 0.250 - \varepsilon 0.750 & 0.612 + \varepsilon 0.612 \\ 0.250 + \varepsilon 1.365 & 0.750 - \varepsilon 0.368 & -0.612 + \varepsilon 0.117 \\ -0.613 + \varepsilon 0.243 & 0.614 + \varepsilon 0.750 & 0.499 - \varepsilon 0.609) \end{bmatrix} \ .$$

The method herein discussed gives

$$\widehat{\theta} = 1.048 + \varepsilon 0.712$$
$$\left\{\widehat{E}\right\} = \left\{ 0.707 + \varepsilon 0.078\ 0.707 - \varepsilon 0.077\ 0.000 + \varepsilon 1.218 \right\}^T$$

**II. Screw parameters for a pseudo rigid body finite motion**

Let us assume that the final coordinates are as follows

$$\{r_1\} = \left\{ 2.600\ 0.380\ 1.500 \right\}^T \ , \quad \{r_2\} = \left\{ 2.800\ 1.130\ 2.100 \right\}^T \ ,$$
$$\{r_3\} = \left\{ 3.300\ -0.330\ 3.700 \right\}^T \ , \quad \{r_4\} = \left\{ 9.000\ 0.960\ 6.300 \right\}^T \ .$$

In this case from (44) and (45) follows:

$$\left[\widehat{A}_1\right] = \begin{bmatrix} 0.780 + \varepsilon 0.016 & 0.200 - \varepsilon 0.689 & 0.640 + \varepsilon 0.443 \\ 0.242 + \varepsilon 1.382 & 0.750 - \varepsilon 0.488 & -0.609 + \varepsilon 0.236 \\ -0.600 + \varepsilon 0.229 & 0.600 + \varepsilon 0.787 & 0.500 - \varepsilon 0.637 \end{bmatrix}$$

$$\left[\widehat{Q}\right] = \begin{bmatrix} 0.770 - \varepsilon 0.141 & 0.221 - \varepsilon 0.741 & 0.599 + \varepsilon 0.456 \\ 0.239 + \varepsilon 1.315 & 0.770 - \varepsilon 0.456 & -0.592 - \varepsilon 0.063 \\ -0.592 + \varepsilon 0.347 & 0.598 + \varepsilon 0.861 & 0.540 - \varepsilon 0.574) \end{bmatrix}$$

$$\left[\widehat{R}\right] = \begin{bmatrix} 1.013 + \varepsilon 0.207 & -0.022 + \varepsilon 0.053 & 0.051 + \varepsilon 0.056 \\ 0.000 + \varepsilon 0.000 & 0.981 - \varepsilon 0.031 & -0.028 + \varepsilon 0.133 \\ 0.000 + \varepsilon 0.000 & 0.000 + \varepsilon 0.000 & 1.013 - \varepsilon 0.175 \end{bmatrix}$$

Assuming $\left[\widehat{A}\right] = \left[\widehat{Q}\right]$ as transform matrix one obtains the finite screw motion parameters

$$\widehat{\theta} = 1.001 + \varepsilon 0.696$$
$$\left\{\widehat{E}\right\} = \left\{ 0.707 + \varepsilon 0.233\ 0.707 - \varepsilon 0.251\ 0.010 + \varepsilon 1.216 \right\}^T$$

**III. Screw parameters for a rigid body infinitesimal motion**

Let

$$\{p_1\} = \{ 1\ 0\ 0 \}^T \ , \quad \{p_2\} = \{ 0\ 1\ 0 \}^T \ , \quad \{p_3\} = \{ 0\ 0\ 1 \}^T \ ,$$

be the position vectors of three points of a rigid body whose velocities are respectively expressed by the following cartesian vectors

$$\{\dot{p}_1\} = \{\, \pi \; 0 \; \sqrt{2} \,\}^T \; , \quad \{\dot{p}_2\} = \{\, 0 \; -\pi \; \sqrt{2} \,\}^T \; , \quad \{\dot{p}_3\} = \{\, \pi \; -\pi \; \sqrt{2} \,\}^T \; .$$

From the application of (50) one obtains

$$\{\widehat{\omega}\} = \left\{ \begin{array}{c} \varepsilon\pi \\ -\varepsilon\pi \\ \pi + \varepsilon\sqrt{2} \end{array} \right\} \; .$$

Hence, the following conclusions are drawn:

- $\{u\} = \{\, 0 \; 0 \; 1 \,\}^T$ is the versor of the infinitesimal screw axis;
- $\omega = \pi$ is the magnitude of the angular velocity;
- $\left\{\, 1 \; 1 \; 0 \,\right\}$ are the coordinates of a point on the screw axis;
- $V = \sqrt{2}$ is the velocity of a point along the screw axis.

## 11 Conclusions

This paper presented several basic algorithms regarding vectors and matrices of dual numbers. The algorithms, arranged in a form suitable for a ready implementation into a code, should provide useful numerical tools for the development of analyses based on the use of dual numbers. In most of the cases the algorithms are accompanied by simple numerical examples to demonstrate their effectiveness.

The paper also included some applications of these algorithms to the solution of common kinematic problems in the field of robotics and biomechanics. In particular two new methods are proposed for the computation of finite and infinitesimal screw motion parameters.

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