Linear Algebra and Numerical Algorithms Using Dual Numbers

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Abstract

Dual number algebra is a powerful mathematical tool for the kinematic and dynamic analysis of spatial mechanisms. With the purpose of exploting new applications, in this paper are presented the dual version of some classical linear algebra algorithms. These algorithms have been tested for the position analysis of the RCCC mechanism and computational improvements over existing methods obtained.

1 Introduction

A *dual number* $\widehat{a}$ is an ordered pair of real numbers (a, a^o) associated with a *real unit* +1, and the *dual unit*, or operator ε, where $\varepsilon^2 = \varepsilon^3 = \ldots = 0$, $0\varepsilon = \varepsilon 0 = 0$, $1\varepsilon = \varepsilon 1 = \varepsilon$. A dual number is usually denoted in the form

$$\widehat{a} = a + \varepsilon a^o \ . \tag{1}$$

The algebra of dual numbers has been originally conceived by W.K. Clifford (1873) [1], but its first applications to mechanics are due to A.P. Kotelnikov (1895)[1] and E. Study (1901) [7].

Dual vector algebra provides a convenient tool for handling mathematical entities such as screws and wrenches. In fact, helicoidal infinitesimal and finite rigid body displacements can be easily composed under the framework of dual vector algebra. Another distinctive feature of dual algebra is conciseness of notation. For these reasons it has been often used for the search of closed form solutions in the field of displacement analysis, kinematic synthesis and dynamic analysis of spatial mechanisms. Applications of dual algebra to the study of the kinematic and dynamic effects of manufacturing and assembly errors in mechanisms are well known (*e.g.* [4, 20, 43, 44, 45, 46]).

One of the purposes of this investigation is the development and implementation of algorithms for the solution of linear algebra problems using dual numbers. The iterative numerical solution of non linear equations is also discussed. Although the algorithms discussed are mainly related to the solution of kinematic problems, we believe they are potentially useful also for dynamic analyses of mechanisms.

The algorithms herein presented can be splitted into the following categories:

1. simple operations involving dual vectors;
2. dual version of basic algorithms of linear algebra;
3. numerical solution of linear and nonlinear equations.

[1] The original paper of A.P. Kotelnikov, published in the Annals of Imperial University of Kazan (1895), is reputed to have been destroyed during the Russian revolution. [61]

The discussed algorithms of linear algebra are very well known in real notation. However, for some of them, namely the QR and SVD decompositions, pseudoinverse and the computation of eigenvalues and eigenvectors, to the best of our knowledge, their dual counterpart has never been discussed. All the algorithms have been implemented using the `Ch` programming language[2] which handles also operations with dual numbers. Some of the routines implemented have been tested on the kinematic analysis of the RCCC spatial linkage and their computational efficiency in the solution of the problem measured. As a result, the proposed strategy for the solution of systems of nonlinear dual number equations seems to offer a substantial reduction of CPU time.

2 Some contributions to mechanics based on dual numbers

Dual numbers and their algebra proved to be a powerful tool for the analysis of mechanical systems. In the following an attempt is made to list recent contributions made in the different fields of mechanics by using dual numbers. No claim of exhaustiveness is made.

Textbooks/monographies entirely dedicated to engineering applications of dual numbers have been authored to F.M Dimentberg [2], R. Beyer [11] and I.S. Fischer [12]. Books with chapters or sections on dual algebra and its applications have been authored by L. Brand [25], M.A. Yaglom [30], J.S. Beggs [62], J. Duffy [29], Gonzáles-Palacios and J. Angeles [26], J. McCarthy [48].

In the following noteworthy contributions of dual algebra to rigid body motion analysis, displacement analysis, kinematic synthesis and dynamic analysis of spatial mechanisms are listed:

- Rigid body motion: A.T. Yang [8], G.R. Veldkamp [6], K. Sugimoto and J. Duffy [21], K. Wohlhart [24];
- Displacement analysis of spatial mechanisms: J. Denavit [13], A.T. Yang [14, 3], A.T. Yang and F. Freudenstein [23], L.M. Hsia and A.T. Yang [54], J. Wittenburg [50], H.H. Cheng [9, 10, 33, 42, 37], I.S. Fischer [16, 15, 18, 12, 38, 39, 40], A.Perez and J.M. McCarthy [69];
- Robotics: You Liang and J.Y.S. Luh.[27], G.R. Pennock and A.T. Yang [31], J. McCarthy [28], M.A. Gonzáles-Palacios et al. [41], G.R. Pennock and K.G. Mattson [32], K. Daniilidis [67];
- Surface shape analysis and computer graphics: P. Azariadis, N. Aspragathos [57], J.A. Schaaf, B. Ravani [64];
- Human body motion analysis: K.K. Teu and W. Kim [58];
- Kinematic synthesis: M.A. Gonzáles-Palacios and J. Angeles [26], J. McCarthy [48], Y.M. Moon and S. Kota [56];
- Dynamic analysis: A.T. Yang [4, 20], C. Bagci [35, 36], M. Keler [5], A.T. Yang and G. Pennock [34], I.S. Fischer and F. Freudenstein [43], C.K. Chen and F. Freudenstein [44], T. Liu and T.W. Lee [72], S.K. Agrawal [55],M. Shoham and V. Brodsky [22], E. Fasse [66], E. Pennestrì et al. [45, 46].

[2]The `Ch` programming language has been developed by Prof. H.H. Cheng of University of California at Davis. More info about its main features are available at the web page `www.softintegration.com`.

3 Dual numbers

Dual numbers can be represented as follows:

- Gaussian representation: $\widehat{a} \equiv a + \varepsilon a^o$.
- Polar representation: $\widehat{a} \equiv \rho\,(1 + \varepsilon t)$, where $\rho = a$ and $t = a^o/a$.
- Exponential representation: $\widehat{a} \equiv \rho \mathrm{e}^{\varepsilon t}$, where $\rho = a$, $t = a^o/a$ and $\mathrm{e}^{\varepsilon t} = 1 + \varepsilon t$.

The adoption of one representation instead of another depends on the context.

4 Dual functions

A function F of a dual variable $\widehat{x} = x + \varepsilon x^o$ can be represented in the form

$$F\left(\widehat{x}\right) = f\left(x, x^o\right) + \varepsilon g\left(x, x^o\right) \;,$$

where f and g are real functions of real variables x and x^o. The necessary and sufficient conditions in order F be analytic are [2]

$$\frac{\partial f}{\partial x^o} = 0\,, \qquad \frac{\partial f}{\partial x} = \frac{\partial g}{\partial x^o}\;. \tag{2}$$

From these follows

$$f\left(\widehat{x}\right) = f\left(x + \varepsilon x^o\right) = f(x) + x^o \frac{\partial f}{\partial x}\;. \tag{3}$$

5 Algebra of dual vectors and matrices

A *line vector* is a vector bound to a definite line $\mathcal{L}$ in space. The *dual vector*

$$\widehat{V} = \vec{v} + \varepsilon \vec{v}^o \tag{4}$$

is combination of two vectors which specifies the position of $\mathcal{L}$ with respect to an arbitrary origin O. The *primary part* is a vector $\vec{v}$ parallel to $\mathcal{L}$ and the *dual part* is $\vec{v}^o = \overrightarrow{OP} \times \vec{v}$, where P is an arbitrary point on $\mathcal{L}$.

5.1 Scalar and cross products of dual vectors

With reference to Figure 1,

$$\widehat{A} = \vec{a} + \varepsilon\left(\vec{r}_1 \times \vec{a}\right)\;,$$
$$\widehat{B} = \vec{b} + \varepsilon\left(\vec{r}_2 \times \vec{b}\right)\;,$$

be two *dual vectors* representing two distinct line vectors and let $\vec{s}^*$ the direction versor of the minimum distance between these line vectors directed from $\vec{a}$ to $\vec{b}$.

In such a context, it is necessary to introduce the concept of *dual angle* [7]

$$\widehat{\theta} = \theta + \varepsilon s \tag{5}$$

as a variable required to characterize the relative position and orientation of line vectors $\widehat{A}$ and $\widehat{B}$. The angle θ is measured counterclockwise about $\vec{s}^*$.

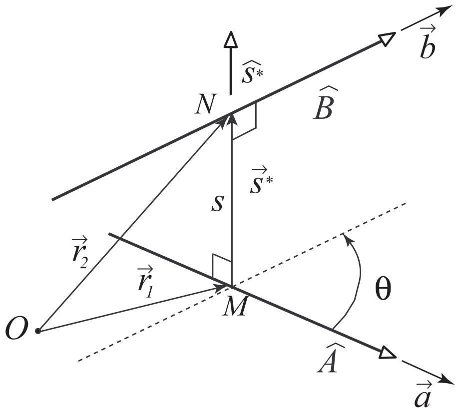

Figure 1: Product of dual vectors:Nomenclature

Table 1: Computational cost of operations with dual numbers

Dual operation	**Mathematical expression**	Mult. and Div.	Sums	Trig. evaluations
Sum	$\widehat{a} \pm \widehat{b} = (a \pm b) + \varepsilon \left(a^o \pm b^o\right)$	-	2	-
Product	$\widehat{a}\widehat{b} = ab + \varepsilon \left(a^o b + a b^o\right)$	3	1	-
Division[1]	$\dfrac{\widehat{a}}{\widehat{b}} = \dfrac{a}{b} + \varepsilon \dfrac{a^o b - a b^o}{b^2}$	5	1	-
Vector scaling	$\widehat{a} \left\{\widehat{n}\right\}$	9	3	-
Dot product[2]	$\left\{\widehat{A}\right\}^T \left\{\widehat{B}\right\} = \{A\}^T \{B\} + \varepsilon \left(\{A^o\}^T \{B\} + \{B^o\}^T \{A\}\right)$	9	7	-
Cross product	$\left[\widetilde{\widehat{A}}\right] \left\{\widehat{B}\right\} = \left[\widetilde{A}\right] \{B\} + \varepsilon \left(\{A\}^T \{B^o\} + \{A^o\}^T \{B\}\right)$	18	12	-
Dual sin	$\sin\widehat{\theta} = \sin\theta + \varepsilon d \cos\theta$	1	0	2
Dual cos	$\sin\widehat{\theta} = \sin\theta + \varepsilon d \cos\theta$	1	0	2
Dual tan	$\tan\widehat{\theta} = \tan\theta + \varepsilon \dfrac{d}{\cos^2\theta}$	2	0	2
Dual tan	$\tan\widehat{\theta} = \tan\theta + \varepsilon \dfrac{d}{\cos^2\theta}$	2	0	2
Dual asin	$\arcsin\widehat{a} = \arcsin a + \varepsilon \dfrac{a^o}{\sqrt{1-a^2}}$	3[3]	1	1
Dual acos	$\arccos\widehat{a} = \arccos a - \varepsilon \dfrac{a^o}{\sqrt{1-a^2}}$	3[3]	1	1
Dual atan	$\arctan\widehat{a} = \arctan a + \varepsilon \dfrac{a^o}{1+a^2}$	2	1	1

[1] Division by a pure dual number εb^o is not defined.
[2] We assume vectors of 3 elements.
[3] The computational cost of a square root operation on real numbers not included in this table is usually considered equivalent to about 8 multiplications/divisions.
[4] $\sim$ denotes a skew-symmetric matrix.

The scalar and cross products of two dual vectors are respectively defined as follows [2]

$$
\begin{aligned}
\widehat{A} \cdot \widehat{B} &= \vec{a} \cdot \vec{b} + \varepsilon \left[\vec{a} \cdot \left(\vec{r}_2 \times \vec{b} \right) + \vec{b} \cdot \left(\vec{r}_1 \times \vec{a} \right) \right] \\
&= \vec{a} \cdot \vec{b} + \varepsilon \left[\left(\vec{r}_1 - \vec{r}_2 \right) \cdot \left(\vec{a} \times \vec{b} \right) \right] \\
&= ab \cos\theta - \varepsilon \left[\left(s\vec{s}^{\star} \right) \cdot \left(ab \sin\theta \vec{s}^{\star} \right) \right] \\
&= ab \left[\cos\theta - \varepsilon s \sin\theta \right] = ab \cos\widehat{\theta} \, . \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
\widehat{A} \times \widehat{B} &= \vec{a} \times \vec{b} + \varepsilon \left[\vec{a} \times \left(\vec{r}_2 \times \vec{b} \right) + \left(\vec{r}_1 \times \vec{a} \right) \times \vec{b} \right] \\
&= \vec{a} \times \vec{b} + \varepsilon \left[\left(\vec{a} \cdot \vec{b} \right) \left(\vec{r}_2 - \vec{r}_1 \right) + \vec{r}_1 \times \left(\vec{a} \times \vec{b} \right) \right] \\
&= ab \left\{ \vec{s}^{\star} \sin\theta + \varepsilon \left[s \cos\theta \vec{s}^{\star} + \sin\theta \left(\vec{r}_1 \times \vec{s}^{\star} \right) \right] \right\} \\
&= ab \widehat{S}^{\star} \left(\sin\theta + \varepsilon s \cos\theta \right) = ab \widehat{S}^{\star} \sin\widehat{\theta} \, , \qquad (7)
\end{aligned}
$$

Table 2 summarizes the result of different dual vectors products for different cases of relative position of line vectors.

Table 2: Noteworthy cases of dual vectors products (Adapted from [2])

Line vector	$\widehat{A} \cdot \widehat{B}$	$\widehat{A} \times \widehat{B}$
Skew	$ab \cos\widehat{\theta}$	$ab\widehat{S}^{\star} \sin\widehat{\theta}$
Incident $(s = 0)$	$ab \cos\theta$	$ab\widehat{S}^{\star}$
Parallel $(\theta = 0)$	ab	$\varepsilon ab\vec{s}^{\star}$
Coaxial $(\theta = s = 0)$	ab	0

5.2 Dual angle between line vectors

The dual angle between the two line vectors

$$
\widehat{A}_i = \vec{a}_i + \varepsilon \left(\vec{s}_i \times \vec{a}_i \right) \, . \quad (i = 1, 2)
$$

must be computed. The computational steps are described in the following and are justified by the geometry depicted in Figure 2.

1. Compute the dual vectors

$$
\widehat{E}_i = \frac{\widehat{A}_i}{\left\| \widehat{A}_i \right\|} \quad (i = 1, 2) \qquad (8)
$$

2. Compute their cross product

$$
\widehat{E}_3 = \frac{\widehat{E}_1 \times \widehat{E}_2}{\left\| \widehat{E}_1 \times \widehat{E}_2 \right\|} \, . \qquad (9)
$$

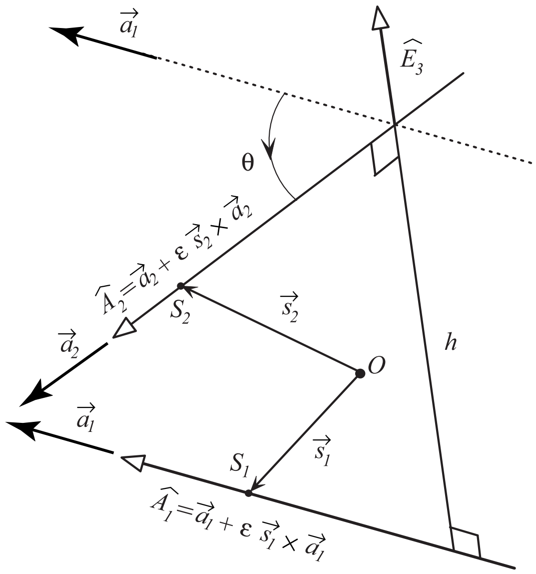

Figure 2: Computation of dual angle between two line vectors

3. Compute cosine and sine of the dual angle $\widehat{\theta}$ between the two line vectors

$$\cos\widehat{\theta} = \widehat{E}_1 \cdot \widehat{E}_2 \ , \tag{10a}$$

$$\sin\widehat{\theta} = \widehat{E}_1 \times \widehat{E}_2 \cdot \widehat{E}_3 \ . \tag{10b}$$

4. Compute dual angle

$$\widehat{\theta} = \mathrm{atan2}\left(\sin\widehat{\theta}, \cos\widehat{\theta}\right) = \theta + \varepsilon h \ . \tag{11}$$

The procedure is not valid if line vectors are parallel. In this case, there is an infinite set of dual vectors $\widehat{E}_3$.

5.3 Sum of two dual vectors

With reference to the geometry of Figure 3, we wish compute the sum

$$\widehat{A} = \widehat{A}_1 + \widehat{A}_2 \tag{12}$$

One can observe that the direction of $\widehat{A}$ is obtained by prescribing a screw motion to $\widehat{A}_1$ defined by the screw axis $\widehat{E}_{12}$ and dual angle $\widehat{\alpha_1}$. On the basis of this observation, the following algorithm can be stated:

1. Compute the dual vectors

$$\widehat{E}_1 = \frac{\widehat{A}_1}{\left\|\widehat{A}_1\right\|} \ ,$$

$$\widehat{E}_2 = \frac{\widehat{A}_2}{\left\|\widehat{A}_2\right\|} \ ,$$

where $\|\cdot\|$ denote the Euclidean norm.

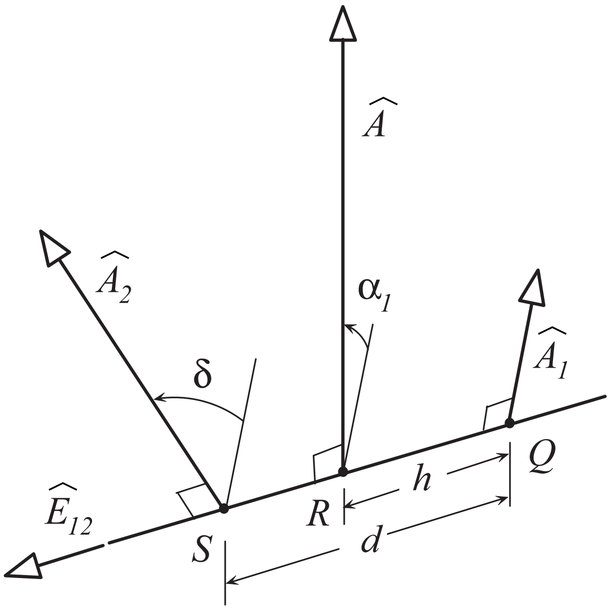

Figure 3: Sum of dual vectors

2. Compute the dual angle $\widehat{\theta}$ and the dual vector $\widehat{E}_{12}$ perpendicular to both $\widehat{A}_1$ and $\widehat{A}_2$ (see previous section).

3. Compute the module of the dual vector sum

$$A = \sqrt{\widehat{A}_1 \cdot \widehat{A}_1 + \widehat{A}_2 \cdot \widehat{A}_2 + 2\widehat{A}_1 \cdot \widehat{A}_2}$$

4. Compute sine and cosine of dual angle $\widehat{\alpha}_1$

$$\sin\widehat{\alpha}_1 = \frac{\sin\widehat{\theta}\sqrt{\widehat{A}_2 \cdot \widehat{A}_2}}{A} \; ,$$
$$\cos\widehat{\alpha}_1 = \frac{\sqrt{\widehat{A}_1 \cdot \widehat{A}_1}}{A} + \sin\widehat{\alpha}_1 \frac{\cos\widehat{\theta}}{\sin\widehat{\theta}} \; ,$$
$$\widehat{\alpha}_1 = \operatorname{atan2}\left(\sin\widehat{\alpha}_1, \cos\widehat{\alpha}_1\right)$$

5. Let

$$\widehat{t} = \tan\frac{\widehat{\alpha}_1}{2}$$

and compute

$$\widehat{T} = \widehat{t}\widehat{E}_{12} \; . \tag{13}$$

6. Apply formula [3]

$$\widehat{E} = \widehat{E}_1 + \frac{2\widehat{T}}{1+\widehat{t}^2} \times \left(\widehat{E}_1 + \widehat{T} \times \widehat{E}_1\right)$$

for the definition of $\widehat{E}_1$ after the screw motion

[3] As will be explained in subsection 6.1.1, this formula represents the extension to dual algebra of the well known Rodrigues' formula.

7. Finally compute

$$\widehat{A} = A\widehat{E} \ .$$

Example: If we assume

$$\widehat{A}_1 = \{ \ 0 \quad 0 \quad 1 \ \}^T \ , \quad \widehat{A}_2 = \{ \ 0 \quad 1 \quad \varepsilon \ \}^T \ ,$$

the previous algorithm give the following numerical values

$$\widehat{\theta} = \frac{\pi}{2} - \varepsilon \ ,$$

$$\widehat{E}_{12} = \{ \ -1 \quad 0 \quad 0 \ \}^T \ ,$$

$$\widehat{\alpha}_1 = \frac{\pi}{4} - \frac{\varepsilon}{2} \ ,$$
$$\widehat{A} = \{ \ 0 \quad 1 \quad 1+\varepsilon \ \}^T \ .$$

The loci formed by all possible $\widehat{A}$ form a ruled surface named *cylindroid*. Ready to use computer-aided solutions of this problem have been recently proposed by A.Perez [70].

5.4 Product of two dual matrices

Assuming that $\left[\widehat{A}\right] = [A] + \varepsilon\,[A^o]$ and $\left[\widehat{B}\right] = [B] + \varepsilon\,[B^o]$ have the appropriate dimensions, then their dual product is defined as follows:

$$\left[\widehat{A}\right]\left[\widehat{B}\right] = [A]\,[B] + \varepsilon\,([A]\,[B^o] + [A^o]\,[B]) \tag{14}$$

5.5 Inverse of a dual matrix

The inverse of a square dual matrix is defined as

$$\left[\widehat{A}\right]\left[\widehat{A}\right]^{-1} = [I] \ . \tag{15}$$

By letting $\left[\widehat{B}\right] = \left[\widehat{A}\right]^{-1}$, from (14) one obtains [27, 49]

$$\left[\widehat{A}\right]^{-1} = [A]^{-1} - \varepsilon\left([A]^{-1}\,[A^o]\,[A]^{-1}\right) \ . \tag{16}$$

5.6 QR decomposition of a dual matrix

A dual matrix $\left[\widehat{A}\right]$ can be decomposed as follows

$$\left[\widehat{A}\right] = \left[\widehat{Q}\right]\left[\widehat{R}\right] \tag{17}$$

where $\left[\widehat{Q}\right] = [Q] + \varepsilon\,[Q^o]$ is an orthogonal matrix and $\left[\widehat{R}\right] = [R] + \varepsilon\,[R^o]$ is an upper triangular matrix.

For the QR decomposition of matrix $\left[\widehat{A}\right] = [A] + \varepsilon\,[A^o]$ the authors could not find a formula for obtaining directly the matrices $\left[\widehat{Q}\right]$ and $\left[\widehat{R}\right]$ as a function of $[A]$ and $[A^o]$. Thus, for the computation of $\left[\widehat{Q}\right]$ and $\left[\widehat{R}\right]$, the modified Gram-Schmidt orthogonalization procedure has been applied to the rows of $\left[\widehat{A}\right]$.

To obtain the QR decomposition one can adapt to dual numbers the modified Gram-Schmidt orthogonalization procedure [65].

Example: The matrix

$$\left[\widehat{A}\right] = \begin{bmatrix} 1+\varepsilon & 2+\varepsilon 3 \\ 3+\varepsilon 9 & 3+\varepsilon \end{bmatrix}$$

can be decomposed in the following matrices[4]:

$$\left[\widehat{Q}\right] = \begin{bmatrix} 0.316-\varepsilon 0.569 & 0.949+\varepsilon 0.190 \\ 0.949+\varepsilon 0.190 & -0.316+\varepsilon 0.569 \end{bmatrix}, \quad \left[\widehat{R}\right] = \begin{bmatrix} 3.162+\varepsilon 8.854 & 3.478+\varepsilon 1.328 \\ 0.000+\varepsilon 0.000 & 0.948+\varepsilon 4.617 \end{bmatrix}.$$

5.7 Pseudoinverse of a dual matrix

Let $\left[\widehat{A}\right]$ be a matrix with m rows and n columns. Its dual Moore-Penrose pseudoinverse is defined as follows:

$$\left[\widehat{A}\right]^{+} = \begin{cases} \left(\left[\widehat{A}\right]^{T}\left[\widehat{A}\right]\right)^{-1}\left[\widehat{A}\right]^{T} & \text{if } m \geq n \\ \left[\widehat{A}\right]^{T}\left(\left[\widehat{A}\right]\left[\widehat{A}\right]^{T}\right)^{-1} & \text{if } m < n \end{cases} \tag{18}$$

If $m > n$, then the left pseudoinverse exists such that $\left[\widehat{A}\right]^{+}\left[\widehat{A}\right] = [I]$. If $m < n$ then the right pseudoinverse exists such that $\left[\widehat{A}\right]\left[\widehat{A}\right]^{+} = [I]$. Adopting a reasoning similar to the one used for the definition of the inverse matrix, one can demonstrate that

$$\left[\widehat{A}\right]^{+} = [A]^{+} - \varepsilon\left([A]^{+}[A^{o}][A]^{+}\right) \tag{19}$$

The definition of the Moore-Penrose matrix is often associated with the least squares solution of a linear system, such as (20), but with the number m of equations different from the number n of unknowns. It is well known that $[A]^{T}[A]$ (or $[A][A]^{T}$) may not have an inverse and, even when it is invertible, one usually obtains large numerical errors from using directly (18). Thus, to obtain significant numerical answers one must use more sophisticated techniques. A summary of efficient algorithms for computing the Moore-Penrose pseudoinverse is reported in [51, 52]. Alternatively, the pseudoinverse matrix can be directly computed from the SVD decomposition.

Example: The Moore-Penrose pseudoinverses of

$$\left[\widehat{A}_1\right] = \begin{bmatrix} 1+\varepsilon 4 & 3+\varepsilon 0 \\ 9+\varepsilon 2 & 22+\varepsilon 4 \\ 4+\varepsilon 4 & 4+\varepsilon 1 \end{bmatrix}, \quad \left[\widehat{A}_2\right] = \begin{bmatrix} 1+\varepsilon 4 & 3+\varepsilon 0 & 4+\varepsilon \\ 9+\varepsilon 2 & 22+\varepsilon 4 & 4+\varepsilon 4 \end{bmatrix}.$$

are, respectively,

$$\left[\widehat{A}_1\right]^{+} = \begin{bmatrix} -0.051+\varepsilon 0.064 & -0.069+\varepsilon 0.082 & 0.418-\varepsilon 0.533 \\ 0.028-\varepsilon 0.025 & 0.073-\varepsilon 0.038 & -0.170+\varepsilon 0.199 \end{bmatrix}, \quad \left[\widehat{A}_2\right]^{+} = \begin{bmatrix} -0.035\varepsilon-0.014 & 0.021+\varepsilon 0.000 \\ -0.038-\varepsilon 0.035 & 0.044-\varepsilon 0.001 \\ 0.287-\varepsilon 0.007 & -0.038-\varepsilon 0.011 \end{bmatrix}.$$

[4]Numerical results are displayed with three decimal digits only.

5.8 Solution of a system of dual linear equations

Let us denote with

$$\left[\widehat{A}\right]\{\widehat{x}\} = \left\{\widehat{b}\right\} \tag{20}$$

a system of linear dual equations [37] where $\left[\widehat{A}\right] = [A] + \varepsilon [A^o]$ and $\left\{\widehat{b}\right\} = \{b\} + \varepsilon \{b^o\}$. Assuming $[A]$ nonsingular. The solution $\{\widehat{x}\} = \{x\} + \varepsilon \{x^o\}$ is computed by solving the systems

$$[A]\{x\} = \{b\} \ , \tag{21a}$$

$$[A]\{x^o\} = \{b^o\} - [A^o]\{x\} \ . \tag{21b}$$

To improve the overall computational efficiency it is convenient to factor matrix $[A]$ only once. Thus, we suggest to proceed as follows:

I) Apply QR decomposition to matrix $[A]$ so that $[A] = [Q][R]$.

II) Solve system $[R]\{x\} = [Q]^T \{b\}$.

III) Solve system $[R]\{x^o\} = [Q]^T (\{b^o\} - [A^o]\{x\})$.

IV) Form the dual vector $\{\widehat{x}\} = \{x\} + \varepsilon \{x^o\}$

Since $[R]$ is an upper triangular matrix, at steps II and III only simple procedures of back substitution are executed.

5.9 Eigenvalues and eigenvectors of a dual matrix

Let us assume that $\left[\widehat{A}\right] = [A] + \varepsilon [A^o]$, with $[A]$ and $[A^o]$ non singular square matrices. The search of eigenvalues $\widehat{\lambda} = \lambda + \varepsilon \lambda^o$ and eigenvectors $\{\widehat{v}\} = \{v\} + \varepsilon \{v^o\}$ requires the solution of the following equation

$$\left[\widehat{A}\right]\{\widehat{v}\} = \widehat{\lambda}\{\widehat{v}\} \ . \tag{22}$$

Expanding the previous equation and splitting into the primary and dual parts we obtain

$$[A - I\lambda]\{v\} = \{0\} \ , \tag{23a}$$

$$\left[\begin{array}{c|c} A - I\lambda & -v \\ \hline v^T & 0 \end{array}\right] \left\{\begin{array}{c} v^o \\ \hline \lambda^o \end{array}\right\} = \left\{\begin{array}{c} -[A^o]\{v\} \\ \hline 0 \end{array}\right\} \ , \tag{23b}$$

after the normalization condition $\{\widehat{v}\}^T \{\widehat{v}\} = 1$ is imposed.

5.10 SVD decomposition of a dual matrix

The dual $m \times n$ matrix $\left[\widehat{A}\right]$ can be decomposed as follows

$$\left[\widehat{A}\right] = \left[\widehat{U}\right]\left[\widehat{\Lambda}\right]\left[\widehat{V}\right]^T \ , \tag{24}$$

where $\left[\widehat{U}\right]^T\left[\widehat{U}\right]=\left[\widehat{V}\right]^T\left[\widehat{V}\right]=\left[\widehat{U}\right]\left[\widehat{U}\right]^T=[I]$ and

$$\left[\widehat{\Lambda}\right]=\begin{bmatrix}\widehat{\lambda}_1 & 0 & 0 & 0\\ 0 & \widehat{\lambda}_2 & 0 & 0\\ 0 & 0 & \ddots & 0\\ 0 & 0 & 0 & \widehat{\lambda}_p\end{bmatrix}, \tag{25}$$

with $0\leq\lambda_1\leq\lambda_2\leq\ldots\leq\lambda_p$, $p=\min(m,n)$. There are many computationally efficient procedures to obtain an SVD decomposition of a matrix [65]. In the case of this investigation, the following has been implemented[5]:

I) Solve the eigenvalue problem

$$\left[\widehat{A}\right]\left[\widehat{A}\right]^T\{\widehat{u}\}=\widehat{\lambda}^2\{\widehat{u}\}\ . \tag{26}$$

II) For each eigenvalue $\widehat{\lambda}_i$ and eigenvector $\{\widehat{u}_i\}$ $i=1,2,\ldots,m$ compute

$$\{\widehat{v}_i\}=\frac{1}{\widehat{\lambda}_i}\left[\widehat{A}\right]^T\{\widehat{u}_i\} \tag{27}$$

III) Form matrices $\left[\widehat{V}\right]$ and $\left[\widehat{U}\right]$ using $\{\widehat{u}_i\}$ and $\{\widehat{v}_i\}$ as columns, respectively.

***Example**:*

Given the matrix

$$\left[\widehat{A}\right]=\begin{bmatrix}1+\varepsilon & 2+\varepsilon 5\\ 2+\varepsilon 0 & 1+\varepsilon\\ 6+\varepsilon 4 & 8+\varepsilon 2\end{bmatrix}$$

the application of the previous computational procedure gives:

$$\left[\widehat{U}\right]=\begin{bmatrix}-0.690-\varepsilon 1.128 & 0.283+\varepsilon 0.288 & 0\\ 0.721-\varepsilon 1.148 & 0.187-\varepsilon 0.008 & 0\\ 0.064+\varepsilon 0.791 & 0.941-\varepsilon 0.085 & 0\end{bmatrix},\quad \left[\widehat{V}\right]=\begin{bmatrix}0.805-\varepsilon 0.050 & 0.593+\varepsilon 0.068 & 0\\ -0.593-\varepsilon 0.068 & 0.805-\varepsilon 0.050 & 0\end{bmatrix},$$

and

$$\left[\widehat{\Lambda}\right]=\begin{bmatrix}1.411+\varepsilon 1.191 & 0 & 0\\ 0 & 10.631+\varepsilon 5.204 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

5.11 Least squares solution of a dual system of linear equations

In kinematics is often required the solution of systems with redundant dual equations. In this case, the least squares solution can be applied. Making use of the SVD decomposition such solution is readily available [65].

[5] If $m>n$ one should apply the procedure to $\left[\widehat{A}\right]^T$ and the matrices $\left[\widehat{U}\right]$ and $\left[\widehat{V}\right]$ reciprocally exchanged.

6 The Principle of Transference

The Principle of Transference proved to be a powerful tool for kinematic analysis of spatial linkages. The Principle has been declared by A.P Kotelnikov for the first time, but the correspondence between equivalent spherical and spatial configurations is due to V.V. Dobrovolski (1947).

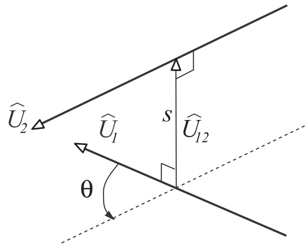

Figure 4: Nomenclature

A thoughtful discussion of the Principle of Transference is given by J. Rooney [61], L.M. Hsia and A.T. Yang [54].

The Principle of Transference can be stated as follows [2, 61, 54]:

> All valid laws and formulae relating to a system of intersecting unit line vectors (and hence involving real variables) are equally valid when applied to an equivalent system of skew vectors, if each variable a, in the original formulae is replaced by the corresponding dual variable $\widehat{a} = a + \varepsilon a^o$.

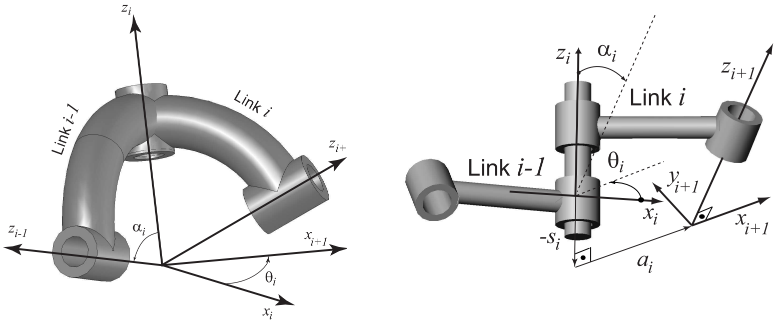

Figure 5: Adjacent links in a spherical mechanism (left) and spatial mechanism (right)

6.1 Applications of the Principle of Transference

In virtue of the Principle of Transference, formulas for the composition of spherical motions can be extended to the general helicoidal motion case by simply substituting the angle of rotation θ with the *dual angle* $\widehat{\theta} = \theta + \varepsilon s$, where s is the displacement of the body along the screw axis.

6.1.1 Extension of the Rodrigues' formula

With reference to the geometry of Figure 6, let $\vec{r}_2$ be the position of vector $\vec{r}_1$ after a rotation about axis of versor $\vec{u}$ of an angle θ.

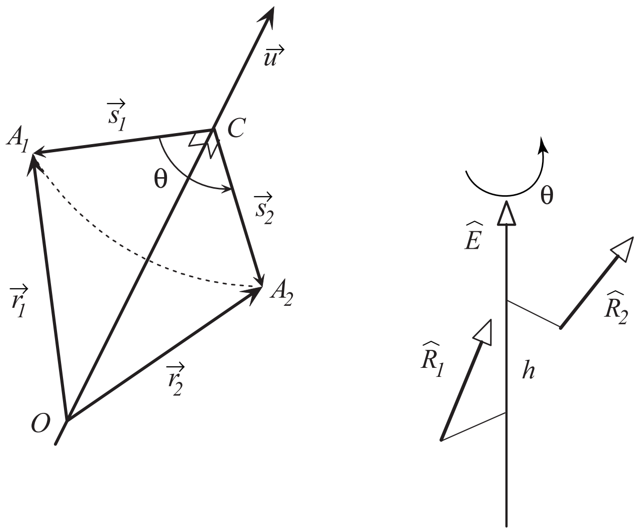

Figure 6: Spherical motion (left) and screw motion about axis h (right)

The well known Rodrigues' formula, in vector notation, can be rewritten in the form

$$\vec{r}_2 = \vec{r}_1 + \frac{2\vec{\tau}}{1+t^2} \times (\vec{r}_1 + \vec{\tau} \times \vec{r}_1) \ , \tag{28}$$

where $t = \tan\frac{\theta}{2}$ and $\vec{\tau} = t\vec{u}$. When $\theta = \pi$ the previous expression cannot be applied and the following should be adopted

$$\vec{r}_2 = 2\,(\vec{u} \cdot \vec{r}_1)\,\vec{u} - \vec{r}_1 \ . \tag{29}$$

By applying the principle of Transference, the Rodrigues' formula (28) can be generalised to define a screw motion about the line vector defined by $\widehat{E}$ as follows

$$\widehat{R}_2 = \widehat{R}_1 + \frac{2\widehat{T}}{1+\tan^2\frac{\widehat{\theta}}{2}} \times \left(\widehat{R}_1 + \widehat{E}\tan\frac{\widehat{\theta}}{2} \times \widehat{R}_1 \right) \ , \tag{30}$$

where

- $\widehat{R}_1$ and $\widehat{R}_2$ are the initial and final positions of a line vector framed to the rigid body, respectively (see Figure 6);
- $\widehat{\theta} = \theta + \varepsilon h$ is the dual angle whose primary part is the angle of rotation and h the displacemnt along h.

6.1.2 Composition of finite screw motions

Given two spherical rotations, the first one of an angle θ_1 about the axis $\vec{u}_1$, and the second of an angle θ_2 about the axis $\vec{u}_2$. Let us introduce the vector

$$\vec{\tau}_i = \tan\frac{\theta_i}{2}\vec{u}_i \quad (i = 1, 2) \; , \tag{31}$$

It can be demonstrated [71, 48] that the resultant spherical motion is defined by the following vector

$$\vec{\tau}_3 = \frac{\vec{\tau}_1 + \vec{\tau}_2 - \vec{\tau}_1 \times \vec{\tau}_2}{1 - \vec{\tau}_1 \cdot \vec{\tau}_2} \; . \tag{32}$$

Consider two finite screw motions about the axes located by the unit line dual vector $\widehat{E}_i$ $(i = 1, 2)$. The rotation angle and the displacement along the axis are denoted with θ_i and h_i $(i = 1, 2)$, respectively. The formula for the composition of these finite motions is obtained by changing the vectors into dual vectors

$$\widehat{T}_3 = \frac{\widehat{T}_1 + \widehat{T}_2 - \widehat{T}_1 \times \widehat{T}_2}{1 - \widehat{T}_1 \cdot \widehat{T}_2} \; , \tag{33}$$

where

$$\widehat{T}_i = \widehat{E}_i \tan\frac{\widehat{\theta}_i}{2} \; , (i = 1, 2, \ldots) \tag{34}$$

6.1.3 The dual form of the Denavit-Hartenberg matrix

With reference to Figure 5, the formulas obtained for the kinematic analysis of spherical mechanisms can be extended to the general spatial mechanism with skew links by substituting:

- the relative position angles θ_i between adjacent links with the dual angles $\widehat{\theta}_i = \theta_i + \varepsilon s_i$;
- the angular length α_i with the dual angle $\widehat{\alpha}_i = \alpha_i + \varepsilon a_i$.

It is worth observing that θ_i, s_i, α_i and a_i form a set of Denavit-Hartenberg (DH) parameters. The dual transform matrix from coordinates system $O_{i+1} - X_{i+1}Y_{i+1}Z_{i+1}$ to $O_i - X_iY_iZ_i$ has the following expression [37]

$$\left[\widehat{A}_i\right] = \left[\widehat{\Theta}_i\right]\left[\widehat{\Lambda}_i\right] \tag{35}$$

where

$$\left[\widehat{\Theta}_i\right] = \begin{bmatrix} \cos\widehat{\theta}_i & -\sin\widehat{\theta}_i & 0 \\ \sin\widehat{\theta}_i & \cos\widehat{\theta}_i & 0 \\ 0 & 0 & 1 \end{bmatrix} , \quad \left[\widehat{\Lambda}_i\right] = \begin{bmatrix} 0 & 0 & 1 \\ \cos\widehat{\alpha}_i & -\sin\widehat{\alpha}_i & 0 \\ \sin\widehat{\alpha}_i & \cos\widehat{\alpha}_i & 0 \end{bmatrix} . \tag{36}$$

7 Nonlinear equations

7.1 Iterative solution of a nonlinear dual equation

Let $F(\widehat{x}) = 0$ be a non linear equation with dual unknown $\widehat{x}$. The monodimensional Newton-Raphson iteration can be written in the form

$$\widehat{x}^{(k+1)} = \widehat{x}^{(k)} - \frac{F\left(\widehat{x}^{(k)}\right)}{F'\left(\widehat{x}^{(k)}\right)} \; , \tag{37}$$

where $F'\left(\widehat{x}^{(k)}\right)$ denotes the derivative $\left.\frac{\partial F}{\partial \widehat{x}}\right|_{\widehat{x}=\widehat{x}^{(k)}}$ and the upperscript k the iteration counter, respectively.

7.2 Iterative solution of a system of nonlinear dual equations

Let $\left\{\widehat{f}\right\} = \left\{ \begin{array}{cccc} \widehat{f}_1 & \widehat{f}_2 & \dots & \widehat{f}_m \end{array} \right\}^T$ be a vector of dual equations in the dual unknowns $\widehat{x}_i$, $(i = 1, 2, \dots, m)$. The Newton-Raphson iteration for the multidimensional case takes the form

$$\{\widehat{x}\}^{(k+1)} = \{\widehat{x}\}^{(k)} - \left[\widehat{J}\right]^{-1} \left\{\widehat{f}\left(\widehat{x}^{(k)}\right)\right\} \ , \tag{38}$$

where $\left[\widehat{J}\right]$ is the dual Jacobian matrix evaluated at $\{\widehat{x}\} = \{\widehat{x}\}^{(k)}$. If the vector of unknowns is formed by both dual and real variables, then the dual pseudoinverse matrix of the jacobian $\left[\widehat{J}\right]^{+}$ must be substituted to $\left[\widehat{J}\right]^{-1}$.

The iteration is halted when $\sum_i (f_i)^2 + (f_i^o)^2 < \epsilon$, where ϵ is a sufficiently small number

8 Applications

Many applications of dual algebra in kinematics require the numerical solution of the system of dual equations of the form

$$\widehat{f}_j\left(\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_n\right) = 0 \ , \quad (j = 1, 2, \dots, m) \tag{39}$$

where $\widehat{x}_i$ $(i = 1, 2, \dots, n)$ are the unknowns.

A common approach (*e.g.* [15, 16, 18]) is to split the equations in the primary and dual parts and then form a system of real equations. In this investigation the possibility of solving the equations *directly* in dual form is explored. H.H. Cheng and S. Thompson [37] followed similar lines. However, during Newton-Raphson iteration, the system of dual equations is splitted into two different linear systems of real equations. The first one composed by the real part of closure equations and the second by their dual part. As demonstrated in this paper, solution schemes can be devised such that the separation in real and dual parts is not required.

In order to test the feasibility of our approach, programs have been written in the Ch programming language [33] for different kinematic analysis applications.

Loop closure equations are often used to solve displacement analysis of spatial mechanisms [19, 17, 59]. By resorting to dual algebra, instead of the classical 4×4 Denavit-Hartenberg transform matrix [19, 17], a 3×3 matrix of dual elements [9, 10] can be used.

8.1 Kinematic analysis of the RCCC spatial mechanism

The following different numerical methods of solution are presented:

- Method A: Iterative numerical solution of a dual nonlinear equation.

- Method B: Iterative solution of system of dual nonlinear equations.

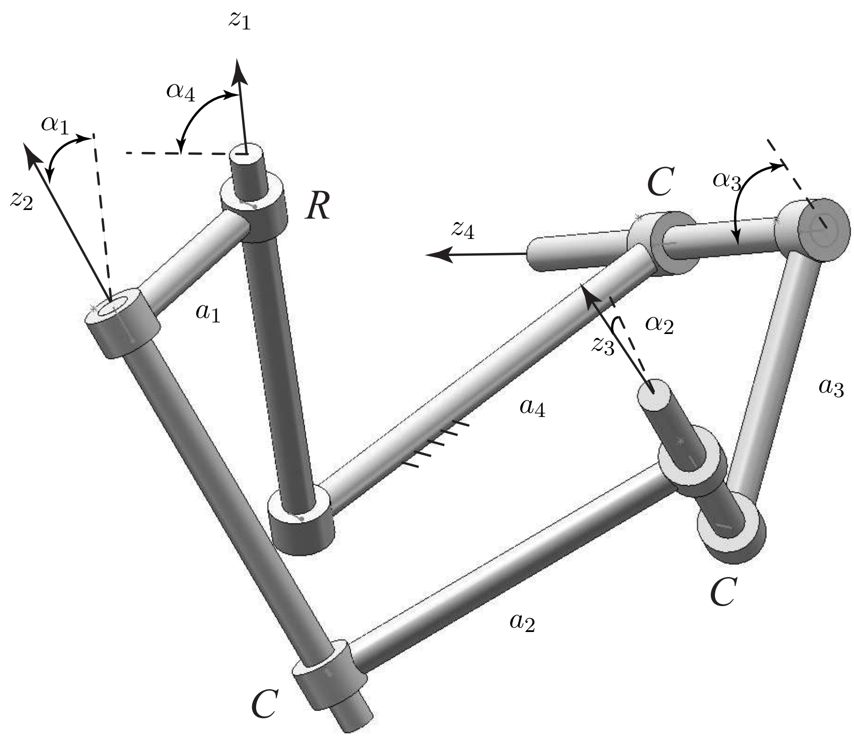

Figure 7: RCCC linkage, R: Revolute pair, C: Cylindrical pair [14, 37]

- Method C: Iterative solution of a system of redundant nonlinear equations.

Method A

The following example is aimed to give a practical application of (37).

In bibliographical references [14, 37] it is demonstrated that position analysis of the RCCC linkage (see Figure 7) reduces to the solution of the following nonlinear equation[6]:

$$\widehat{F}\left(\widehat{\theta}_4\right) \equiv \widehat{A}\sin\widehat{\theta}_4 + \widehat{B}\cos\widehat{\theta}_4 + \widehat{C} = 0 \,, \tag{40}$$

where

$$\widehat{A} = \sin\widehat{\alpha}_1 \sin\widehat{\alpha}_3 \sin\widehat{\theta}_1 \,, \tag{41}$$

$$\widehat{B} = -\sin\widehat{\alpha}_3\left(\cos\widehat{\alpha}_1 \sin\widehat{\alpha}_4 + \sin\widehat{\alpha}_1 \cos\widehat{\alpha}_4 \cos\widehat{\theta}_1\right) \,, \tag{42}$$

$$\widehat{C} = \cos\widehat{\alpha}_3\left(\cos\widehat{\alpha}_1 \cos\widehat{\alpha}_4 - \sin\widehat{\alpha}_1 \sin\widehat{\alpha}_4 \cos\widehat{\theta}_1\right) - \cos\widehat{\alpha}_2 \,. \tag{43}$$

We acknowledge that (40) can be solved by substituting the $\sin\widehat{\theta}_4$ and $\cos\widehat{\theta}_4$ functions with their $\tan\frac{\widehat{\theta}_4}{2}$ correspondent. However, the numerical solution of such equation is herein reported for demonstration purposes only. If the link dimensions, in term of DH parameters, are as follows: $\alpha_1 = 30°$, $\alpha_2 = 55°$, $\alpha_3 = 45°$, $\alpha_4 = 60°$, a_1 = 2, a_2 = 4, a_3 = 3, a_4 = 5, then we have

$$\widehat{A} = 0.227260 + \varepsilon 1.469030 \,, \quad \widehat{B} = -0.665749 - \varepsilon 2.212148 \,, \quad \widehat{C} = -0.501943 - \varepsilon 1.433104 \,.$$

Assuming $\widehat{\theta}_4^{(0)} = 1.745329 - \varepsilon 1.300000$ as solution guess, Table 3 reports the results of the Newton-Raphson iteration.

[6]The nomenclature adopted in this section follows the one of reference [37].

Table 3: Results of dual monodimensional Newton-Raphson iteration applied to equation (40).

k	$\widehat{\theta}_4^{(k)}$	$\widehat{F}\left(\widehat{\theta}_4^{(k)}\right)$
0	1.745329 -ε 1.300000	-0.162529 -ε 0.403280
1	2.009102, -ε1.657790	-0.013617 - ε 0.003539
2	2.035995 -ε 1.767060	-0.000178 + ε 0.000931
3	2.036356 -ε 1.770564	-0.000000 + ε 0.000001

Method B

The mechanism matrix closure equation can be written in the form

$$\left[\widehat{\Lambda}_3\right]\left[\widehat{\Theta}_4\right]\left[\widehat{\Lambda}_4\right]\left[\widehat{\Theta}_1\right]\left[\widehat{\Lambda}_1\right]-\left[\widehat{\Theta}_3\right]^T\left[\widehat{\Lambda}_2\right]^T\left[\widehat{\Theta}_2\right]^T=[0]\ . \tag{44}$$

Only three of the nine elements of the above matrix form a set of independent equations[7]. If we choose the elements (3,3), (3,1), (1,1), the system of nonlinear equations is the following $\left\{\widehat{f}\right\}\equiv\left\{\begin{array}{ccc}\widehat{f}_1 & \widehat{f}_2 & \widehat{f}_3\end{array}\right\}^T=\{0\}$, where

$$\begin{aligned}\widehat{f}_1=&\left[\sin\widehat{\alpha}_3\sin\widehat{\theta}_4\sin\widehat{\theta}_1-\left(\sin\widehat{\alpha}_3\cos\widehat{\alpha}_4\cos\widehat{\theta}_4-\cos\widehat{\alpha}_3\sin\widehat{\alpha}_4\right)\cos\widehat{\theta}_1\right]\sin\widehat{\alpha}_1\\&+\left(\cos\widehat{\alpha}_3\cos\widehat{\alpha}_4-\sin\widehat{\alpha}_3\cos\widehat{\theta}_4\sin\widehat{\alpha}_4\right)\cos\widehat{\alpha}_1-\cos\widehat{\alpha}_2\ ,\end{aligned} \tag{45a}$$

$$\widehat{f}_2=\sin\widehat{\alpha}_3\sin\widehat{\theta}_4\cos\widehat{\theta}_1+\left(\sin\widehat{\alpha}_3\cos\widehat{\alpha}_4\cos\widehat{\theta}_4+\cos\widehat{\alpha}_3\sin\widehat{\alpha}_4\right)\sin\widehat{\theta}_1-\sin\widehat{\alpha}_2\sin\widehat{\theta}_2\ , \tag{45b}$$

$$\widehat{f}_3=\cos\widehat{\theta}_4\cos\widehat{\theta}_1-\cos\widehat{\alpha}_4\sin\widehat{\theta}_4\sin\widehat{\theta}_1-\cos\widehat{\theta}_3\cos\widehat{\theta}_2+\sin\widehat{\theta}_3\cos\widehat{\alpha}_2\sin\widehat{\theta}_2\ . \tag{45c}$$

Assuming the same link dimensions specified in the previous example, the results of iteration (38) are summarized in Table 4.

The total number of floating point operations required for the evaluation of (45) is as follows: 110 multiplications, 46 sums and 76 trigonometric functions. Adopting the method of constraint described in the tetxbook of E.J. Haug [74] 21 scalar equations would be involved in the analysis. The evaluation of these equations requires the following floating point operations: 328 multiplications, 213 sums and 6 trigonometric functions.

Method C

The solution method of matrix loop closure equations, originally proposed by J.J. Uicker *al.* [63], can be adapted to dual equations (*e.g.* [37, 18]). For completeness the method is summarized with reference to the position analysis of the RCCC linkage. If $\left\{\widehat{\theta}^{(k)}\right\}$ is a solution guess vector and $\left\{\Delta\widehat{\theta}^{(k)}\right\}$ the

[7] The choice of independent equations must obey the rules stated by the following theorem ([62], p.5):

> The orientation or attitude of a Cartesian system of coordinates 2 relative to system 1 is uniquely specified by stating the values of three elements of the transform matrix which lie in any two rows, and one of four possible values for a fourth element, chosen so that the four elements do not lie in the same minor, and less than three elements in a row. The word *row* may be replaced by the word *column* throughout the theorem.

Table 4: Results of dual multidimensional Newton-Raphson iteration applied to system of equations (45).

k	$\widehat{\theta}_4^{(k)}$	$\sum_{i=1}^{3} (f_i)^2 + (f_i^o)^2$
0	1.745329 -ε 1.300000	0.552520
1	2.009102 -ε 1.657790	0.002172
2	2.035995 - ε 1.767061	0.205704
3	2.036356 - ε 1.770565	0.002170
4	2.036356 - ε 1.770568	0.000049
5	2.036356 -ε 1.770567	0.000003

corresponding correction vector, the following equality can be established:

$$\left[\widehat{A}_i\left(\widehat{\theta}_i^{(k)} + \Delta\widehat{\theta}_i^{(k)}\right)\right] = \left[\widehat{A}_i^{(k)}\right] + \left[\left.\frac{\partial A_i^{(k)}}{\partial \theta_i}\right|_{\widehat{\theta}_i = \widehat{\theta}_i^{(k)}}\right] \Delta\widehat{\theta}_i^{(k)} \tag{46}$$

The partial derivative is also expressed by the product

$$\left[\left.\frac{\partial A_i^{(k)}}{\partial \theta_i}\right|_{\widehat{\theta}_i = \widehat{\theta}_i^{(k)}}\right] = [Q]\left[\widehat{A}_i^{(k)}\right] \quad (i = 2, 3, 4) \tag{47}$$

where

$$[Q] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{48}$$

Hence, when in the the matrix loop equation

$$\left[\widehat{A}_1\left(\widehat{\theta}_1\right)\right]\left[\widehat{A}_2\left(\widehat{\theta}_2\right)\right]\left[\widehat{A}_3\left(\widehat{\theta}_3\right)\right]\left[\widehat{A}_4\left(\widehat{\theta}_4\right)\right] = [I] \ . \tag{49}$$

we substitute the equalities

$$\widehat{\theta}_i = \widehat{\theta}_i^{(k)} + \Delta\widehat{\theta}_i^{(k)} \quad (i = 2, 3, 4) \tag{50}$$

we obtain, neglecting higher order terms,

$$[B_2]\,\Delta\widehat{\theta}_2^{(k)} + [B_3]\,\Delta\widehat{\theta}_3^{(k)} + [B_4]\,\Delta\widehat{\theta}_4^{(k)} = [I] - [B_1] \tag{51}$$

where

$$\left[\widehat{B}_1\right] = \left[\widehat{A}_1\right]\left[\widehat{A}_2^{(k)}\right]\left[\widehat{A}_3^{(k)}\right]\left[\widehat{A}_4^{(k)}\right] \tag{52}$$

and

$$\left[\widehat{B}_i\right] = \left[\widehat{A}_1\right] \ldots \left[\widehat{A}_{i-1}^{(k)}\right][Q]\left[\widehat{A}_{i+1}^{(k)}\right] \ldots \left[\widehat{A}_4^{(k)}\right] \ , (i = 2, 3, 4) \ . \tag{53}$$

The system of redundant but consistent equations (51) can be rewritten in the following ready for computation form

$$\begin{bmatrix} \widehat{B}_{2_{11}} & \widehat{B}_{3_{11}} & \widehat{B}_{4_{11}} \\ \widehat{B}_{2_{21}} & \widehat{B}_{3_{21}} & \widehat{B}_{4_{21}} \\ \widehat{B}_{2_{31}} & \widehat{B}_{3_{31}} & \widehat{B}_{4_{31}} \\ \widehat{B}_{2_{22}} & \widehat{B}_{3_{22}} & \widehat{B}_{4_{22}} \\ \widehat{B}_{2_{32}} & \widehat{B}_{3_{32}} & \widehat{B}_{4_{32}} \\ \widehat{B}_{2_{33}} & \widehat{B}_{3_{33}} & \widehat{B}_{4_{33}} \end{bmatrix} \left\{ \begin{array}{c} \Delta\widehat{\theta}_2^{(k)} \\ \Delta\widehat{\theta}_3^{(k)} \\ \Delta\widehat{\theta}_4^{(k)} \end{array} \right\} = \left\{ \begin{array}{c} 1 - \widehat{B}_{1_{21}} \\ -\widehat{B}_{1_{21}} \\ -\widehat{B}_{1_{31}} \\ 1 - \widehat{B}_{1_{22}} \\ -\widehat{B}_{1_{32}} \\ 1 - \widehat{B}_{1_{33}} \end{array} \right\} \tag{54}$$

With obvious substitutions, this matrix equation can be more concisely expressed as

$$\left[\widehat{M}\right] \left\{\Delta\widehat{\theta}^{(k)}\right\} = \{\widehat{v}\} \ . \tag{55}$$

There are different alternatives to solve (55). Adopting the least-squares method, the correction vector is obtained by solving the linear system

$$\left[\widehat{M}\right]^T \left[\widehat{M}\right] \left\{\Delta\widehat{\theta}^{(k)}\right\} = \left[\widehat{M}\right]^T \{\widehat{v}\} \ . \tag{56}$$

of three dual equations in three dual unknowns. As previously explained, there are two different computing schemes. Otherwise, one can iteratively apply the formula (50), where the pseudoinverse matrix of $\left[\widehat{M}\right]$ is used to compute the correction vector

$$\left\{\Delta\widehat{\theta}^{(k)}\right\} = [M]^+ \{v\} \ . \tag{57}$$

From the numerical point of view, the solution method C can be executed in the following three different alternatives:

- Alternative C1: Solve the linear system (56) using the equations (21), as suggested in [37]. The solution is herein obtained by means of the standard Ch numerical library procedure `linsolve`.
- Alternative C2: Solve the linear system (56) using the alternative procedure proposed in this paper for the solution of dual linear equations system (see end of subsection 5.8).
- Alternative C3: Solve the linear system (56) making use of the dual QR decomposition (see subsection 5.6).
- Alternative C4: Solve the redundant equations system (55) computing the pseudoinverse matrix (see subsection 5.7) and making use of equation (57).

All the alternatives listed above have been implemented in the Ch programming language and the CPU time[8] required by each of them reported in Table 5.

[8]Under Windows XP operating system it is not usually possible the precise monitoring of CPU time elapsed during the run of a task. For this reason, each alternative has been executed 10 times and the average CPU time reported.

Table 5: Comparison of CPU times required for the analysis of the RCCC spatial linkage

Alternative	CPU Time (s)
C1	1.422
C2	1.214
C3	1.175
C4	1.276

9 Conclusions

This paper presented several basic algorithms regarding vectors and matrices of dual numbers. The algorithms, arranged in a form suitable for a ready implementation into a code, should provide useful numerical tools for the development of analyses based on the use of dual numbers. In most of the cases the algorithms are accompanied by simple numerical examples to demonstrate their effectiveness. The algorithms of pseudoinverse, eigenvalue and SVD decomposition are believed to be novel in the field of dual numbers. The availability of such algorithms should broad the field of application of dual numbers. The paper also included some applications of these algorithms to the solution of classical kinematic problems. In particular different approaches to the numerical kinematic analysis of the RCCC spatial mechanism have been discussed and compared on the basis of their computational efficiency. The proposed approach for the solution of a redundant system of nonlinear equations offers a computational gain with respect other approaches. All `Ch` routines implemented are available upon request.

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