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COORDINATES REDUCTION AND NUMERICAL INTEGRATION OF MODELS OF CONSTRAINED MULTIBODY SYSTEMS

F. Del Citto, E. Pennestrì, and L. Vita

Department of Mechanical Engineering
Via del Politecnico 1, 00133 Rome, University of Rome Tor Vergata, Italy
e-mails: delcitto@tin.it, pennestri@mec.uniroma2.it and vita@ing.uniroma2.it

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Abstract. *The authors present a variation of the constraint orthogonalization method proposed by Kim and Vanderploeg [1] for constrained multibody systems. Firstly the theoretical bases of the QTZ decomposition are presented, then it is reported how to apply this methodology to reduce the constraints number and define the independent velocities.*

A section is dedicated to three different solution methods based upon the QTZ decomposition. The first one leads to the reduced system of equation of motion, the other two methods allow to rearrange the Differential-Algebraic-Equations system into state space form by means of independent accelerations and a minimum set of coordinates, respectively. Finally two numerical examples based on the methodology herein proposed are reported. The results obtained by means of the in house developed multibody code NumDyn3D, based on QTZ decomposition, are compared to the ones obtained by means of the commercial software Working Model.

1 INTRODUCTION

Constraint orthogonalization is a well known technique in the dynamic simulation of multibody systems with redundant constraints [1, 2, 3]. This paper discuss the use of the QTZ decomposition for the numerical integration of DAE systems. A variation of the well known method of constraint orthogonalization due to Kim and Vanderploeg is introduced. Moreover three new different formulations are presented and their accuracy and execution times compared with those of a commercial software.

2 CONSTRAINED EQUATION OF MOTIONS

In a multibody formulation with redundant coordinates, by joining the differential equations of motion for the system of rigid bodies and the positional kinematic constraints, one obtains the following set of of Differential-Algebraic-Equations (DAE)

$$\begin{cases} [M]\ \{\ddot{q}\} + [\Psi_q]^T\ \{\lambda\}\ =\ \{g\} \\ \{\Psi\,(q,t)\} = \{0\} \end{cases} \tag{1}$$

where $[M]$ is the inertia matrix, $\{\lambda\}$ is a vector of unknown Lagrangian multipliers, $[\Psi_q]$ is the constraints Jacobian matrix, $\{\Psi\,(q,t)\}$ is the vector of kinematic constraints and $\{g\}$ is the vector of external generalized forces.

The numerical integration of (1) requires, at $t = t_0$, the initial conditions

$$\begin{cases} \{q\,(t_0)\} = \{q_0\} \\ \{\dot{q}\,(t_0)\} = \{\dot{q}_0\} \end{cases} \tag{2}$$

consistent with the kinematic constraints.

3 THEORETICAL BASES OF THE QTZ DECOMPOSITION

Let $[\Psi_q]$ be a $(m \times n)$ matrix with rank [1] $r \leqslant \min(m,n)$. It is always possible to rewrite the transpose Jacobian matrix by a QR decomposition with column pivoting as

$$[\Psi_q]^T\,[P]\ =\ [Q]\,[R] \tag{3}$$

where $[P]$ is a $(m \times m)$ permutation matrix [2] of the columns of $[\Psi_q]^T$, $[Q]$ is a $(n \times n)$ orthogonal matrix, while $[R]$ is a $(n \times m)$ whose structure may vary. When $r \leqslant m \leqslant n$, the matrix $[R]$ has the following structure

$$[R]\ =\ \begin{bmatrix} [R_{11}] & [R_{12}] \\ [R_{21}] & [R_{22}] \\ [0] & [0] \end{bmatrix} \tag{4}$$

where [4]:

[1]The method presented herein allows the system to have any number of degrees-of-freedom and redundant constraints.

[2]By means of this permutation matrix the rows of $[R]$ are ordered so that $|r_{11}| \geq |r_{22}| \geq \cdots \geq |r_{nn}|$

- $[R_{11}]$ upper triangular $(r \times r)$ matrix
- $[R_{21}]$ exactly null $(m-r) \times r$ matrix
- $[R_{22}]$ numerically null $(m-r) \times (m-r)$ matrix
- $[R_{12}]$ not null $r \times (m-r)$ matrix.

When $r \leqslant n < m$, the matrix $[R]$ has the following structure:

$$[R] = \begin{bmatrix} [R_{11}] & [R_{12}] \\ [R_{21}] & [R_{22}] \end{bmatrix} \tag{5}$$

where:

- $[R_{11}]$ upper triangular $(r \times r)$ matrix
- $[R_{21}]$ exactly null $(n-r) \times r$ matrix
- $[R_{22}]$ numerically null $(n-r) \times (m-r)$ matrix
- $[R_{12}]$ not null $r \times (m-r)$ matrix

A fast and reliable way to compute the rank of the Jacobian matrix is to count the elements of the main diagonal of $[R]$ *significantly* different from zero.

In a more compact notation, the QR decomposition of the transpose Jacobian matrix takes the form

$$[\Psi_q]^T [P] = [Q] \begin{bmatrix} [R'] \\ [0] \end{bmatrix} \tag{6}$$

where $[R']$ is an upper trapezoidal $(r \times m)$ matrix, while the dimensions of the null block[3] are $(n-r) \times m$.

The upper trapezoidal matrix $[R']$ can be decomposed as follows

$$[R'] = \big[[R_{11}] \ [R_{12}]\big] = \big[[T] \ [0]\big] [Z] \tag{7}$$

where $[T]$ is a full rank upper triangular $(r \times r)$ matrix, while $[Z]$ is an orthogonal $(m \times m)$ matrix.

The matrix $[R_{11}]$ is never ill–conditioned.

For our purposes, the transpose Jacobian matrix is decomposed as follows:

$$[\Psi_q]^T [P] = [Q] \begin{bmatrix} [T] & [0] \\ [0] & [0] \end{bmatrix} [Z] \tag{8}$$

[3]This block contains both exactly null and numerically null entries.

This decomposition can be applied to all matrices and is independent from their rank. If the matrix has full column rank, i.e. $r = m$, and the mechanical system has at least one degree-of-freedom, then $[Z] = [I]_{m \times m}$.

More in detail, it is possible to identify two different blocks within $[Q]$, i.e.:

$$[\Psi_q]^T [P] = \left[[Q_1] \;\; [Q_2] \right] \begin{bmatrix} [T] & [0] \\ [0] & [0] \end{bmatrix} [Z] \tag{9}$$

where $[Q_1]$ is a $(n \times r)$ block, while $[Q_2]$ is a $n \times (n - r)$ block.

3.1 The properties of $[Q]$

Equation (9) can be rewritten as

$$[\Psi_q]^T [P] [Z]^T = \left[[Q_1] \;\; [Q_2] \right] \begin{bmatrix} [T] & [0] \\ [0] & [0] \end{bmatrix} = \left[[Q_1] [T] \;\; [0] \right] \tag{10}$$

or

$$[\Psi_q]^T [P] = \left[[Q_1] [T] \;\; [0] \right] [Z] \tag{11}$$

Since $[Q_1]^T [Q_1] = [I]$, this shows that, regardless of the rigid orthogonal transformation produced by $[Z]$, the columns of $[Q_1]$ form an orthonormal base of the column space of $[\Psi_q]^T$. Moreover, premultiplication of equation (11) by $[Q_2]^T$ gives

$$[Q_2]^T [\Psi_q]^T [P] = [Q_2]^T \left[[Q_1] [T] \;\; [0] \right] [Z] = [0] \tag{12}$$

or

$$[\Psi_q] [Q_2] = [0] \tag{13}$$

Equation (13) shows that the columns of $[Q_2]$ form a base for the Jacobian matrix kernel.

For any given $[P]$ and $[Z]$,it is possible to show, through the unicity of the Cholesky decomposition, that $[Q_1]$ matrix is unique [1, 7], while $[Q_2]$ has the following properties

$$[Q_2]^T [Q_1] = [0] \tag{14a}$$

$$[Q_2]^T [Q_2] = [I] \tag{14b}$$

3.2 The reduction of the constraints number

In this section is described the procedure for removing the redundant constraints. The dependent constraints, detected by the QR decomposition with column pivoting, can be removed. In fact, rearranged the Jacobian matrix in the form

$$\left[\Psi'_q \right]^T \equiv [\Psi_q]^T [P] [Z]^T \tag{15}$$

from equation (9) one obtains

$$\left[\Psi'_q\right]^T = \left[[Q_1]\ [Q_2]\right] \begin{bmatrix} [T] & [0] \\ [0] & [0] \end{bmatrix} = \left[[Q_1]\,[T]\ [0]\right] \tag{16}$$

that is, by means of the orthogonal transformation on the rearranged Jacobian matrix, the last $(m-r)$ rows of $\left[\Psi'_q\right]$ can be zeroed.

The $(m \times n)$ matrix $\left[\Psi'_q\right]$ can therefore be further partitioned

$$\left[\Psi'_q\right] = \begin{bmatrix} \left[\Psi''_q\right] \\ [0] \end{bmatrix} \tag{17}$$

where:

- $\left[\Psi''_q\right]$, $(r \times n)$ matrix, identify a Jacobian of independent *constraints*
- $[0]$ is a $(m-r) \times n$ null elements matrix.

It follows that an useful way to write the complete QTZ decomposition is

$$\left[\Psi''_q\right]^T = \left[[Q_1]\ [Q_2]\right] \begin{bmatrix} [T] \\ [0] \end{bmatrix} \tag{18}$$

From equation (18), it follows immediately that

$$\left[\Psi''_q\right]^T = [Q_1]\,[T] \tag{19}$$

$$\left[\Psi''_q\right]\,[Q_2] = [0] \tag{20}$$

3.3 Definition of independent velocities

Assume that the generalized coordinates vector $\{q_0\}$ at time t_0 satisfy the constraint equations

$$\left\{\Psi\big(\{q_0\}\,, t_0\big)\right\} = \{0\} \tag{21}$$

The first time derivative of equation (21) gives

$$\left[\Psi_q\big(\{q_0\}\,, t_0\big)\right]\{\dot{q}_0\} = \{b_0\} \tag{22}$$

where

$$\{b_0\} = -\left\{\Psi_t\big(\{q_0\}\,, t_0\big)\right\} \tag{23}$$

Premultiplication of (22) times $[Z]\,[P]^T$, as defined before, gives

$$[Z]\,[P]^T \left[\Psi_q\big(\{q_0\}\,, t_0\big)\right]\{\dot{q}_0\} = [Z]\,[P]^T\{b_0\} \tag{24}$$

which, taking into account equation (15), can be rewritten in the following form

$$\left[\Psi_q'\big(\{q_0\}\,,t_0\big)\right]\{\dot{q}_0\} \;=\; \{b_0'\} \tag{25}$$

where

$$\{b_0'\} = [Z]\,[P]^T\,\{b_0\} \tag{26}$$

More in detail, the vector $\{b_0'\}$ is made of two distinct parts. In fact

$$\left[\Psi_q'\big(\{q_0\}\,,t_0\big)\right]\{\dot{q}_0\} \;=\; \begin{bmatrix} \left[\Psi_q''\big(\{q_0\}\,,t_0\big)\right] \\ [0] \end{bmatrix}\{\dot{q}_0\} \;=\; \{b_0'\} \;=\; \left\{ \begin{array}{c} \{b_0''\} \\ \{0\} \end{array} \right\} \tag{27}$$

where $\{b_0''\}$ is a vector of length r.

This yields

$$\left[\Psi_q''\big(\{q_0\}\,,t_0\big)\right]\{\dot{q}_0\} \;=\; \{b_0''\}\;, \tag{28}$$

whit Ψ_q'' matrix with full rank.

Inserting equation (19) into equation (28) gives

$$[Q_1]^T\{\dot{q}_0\} \;=\; \left([T]^T\right)^{-1}\{b_0''\} \;=\; \left\{\widehat{b}_0''\right\}^4 \tag{29}$$

The general solution of equation (28) is a sum of a particular and a homogeneous solution (see Appendix).

$$\{\dot{q}_0\} \;=\; \{\dot{q}_0'\} + \{\dot{q}_0''\} \;=\; [Q_1]\left\{\widehat{b}_0''\right\} + [Q_2]\,\{\dot{z}\} \tag{30}$$

where $\{\dot{z}\}$ is a vector of freely prescribed velocities with order $n-r$.

In order to maintain directional continuity of $[Q_2]$ within one ore more time steps, in analogy to what proposed by Kim and Vanderploeg [1], but referring to the reduced Jacobian Matrix $\left[\Psi_q''\right]$, the matrices

$$\left[\widehat{Q}_1\right] \;=\; [Q_1]\left(\left[\Psi_q''\right][Q_1]\right)^{-1} \tag{31}$$

$$\left[\widehat{Q}_2\right] \;=\; [Q_2] - \left[\widehat{Q}_1\right]\left[\Psi_q''\right][Q_2] \tag{32}$$

are defined. It is now possible to introduce the following relationship between velocities as[5] :

$$\{\dot{q}\} \;=\; \left[\left[\widehat{Q}_1\right]\;\left[\widehat{Q}_2\right]\right]\left\{ \begin{array}{c} \{b''\} \\ \{\dot{z}\} \end{array} \right\} \tag{33}$$

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[4]The vector $\left\{\widehat{b}_0''\right\}$ can be easily obtained without inverting $[T]^T$ solving the linear system $[T]^T\left\{\widehat{b}_0''\right\} = \{b_0''\}$.

[5]For a comprehensive discussion about definition, properties and use of $\left[\widehat{Q}\right]$ see [7]

Two main properties of matrix $\left[\widehat{Q}\right]$ should be observed:

$$\left[\Psi_q''\right]\left[\widehat{Q}_1\right] = \left[\Psi_q''\right][Q_1]\left(\left[\Psi_q''\right][Q_1]\right)^{-1} = [I]_{r\times r} \tag{34}$$

$$\left[\Psi_q''\right]\left[\widehat{Q}_2\right] = \left[\Psi_q''\right][Q_2] - \left[\Psi_q''\right]\left[\widehat{Q}_1\right]\left[\Psi_q''\right][Q_2] = [0]_{r\times(n-r)} \tag{35}$$

3.3.1 Acceleration relation

From equations (26) and (27), equation (33) can be rewritten as

$$\{\dot{q}\} = -\left[\widehat{Q}_1\right]\left[\bar{I}\right][Z][P]^T\{\Psi_t\} + \left[\widehat{Q}_2\right]\{\dot{z}\} \tag{36}$$

where $\left[\bar{I}\right]$ is a $(r \times n)$ matrix having the elements of the main diagonal set to 1, while the remaining are all null.

The time derivative of this last equation gives

$$\begin{aligned}
\frac{d}{dt}\{\dot{q}\} &= \frac{d}{dt}\left(-\left[\widehat{Q}_1\right]\left[\bar{I}\right][Z][P]^T\{\Psi_t\} + \left[\widehat{Q}_2\right]\{\dot{z}\}\right)\\
&= \left[\widehat{Q}_1\right]\left[\bar{I}\right][Z][P]^T\frac{d}{dt}\left(-\{\Psi_t\}\right) + \left[\widehat{Q}_2\right]\frac{d}{dt}\{\dot{z}\}\\
&= \left[\widehat{Q}_1\right]\left[\bar{I}\right][Z][P]^T\left(-[\Psi_{qt}]\{\dot{q}\} - \{\Psi_{tt}\}\right) + \left[\widehat{Q}_2\right]\{\ddot{z}\}
\end{aligned}$$

which, letting

$$\{a\} = -[\Psi_{qt}]\{\dot{q}\} - \{\Psi_{tt}\}\ , \tag{37}$$

$$\{a''\} = \left[\bar{I}\right][Z][P]^T\{a\}\ , \tag{38}$$

can be rewritten as

$$\{\ddot{q}\} = \left[\left[\widehat{Q}_1\right]\ \left[\widehat{Q}_2\right]\right]\left\{\begin{array}{c}\{a''\}\\ \{\ddot{z}\}\end{array}\right\} \tag{39}$$

where $\{\ddot{z}\}$ is the vector of the independent accelerations. Its dimension equals the number $(n-r)$ of degrees-of-freedom of the system.

3.4 Solving the equations of motion

Three different solution methods, based upon the QTZ decomposition are herein discussed. The first one leads to automatically eliminate dependent rows in the Jacobian. Thus the resulting coupled set of DAE can be solved. The other two methods, by means of independent accelerations or by finding a transformed minimum set of coordinates, reduce the DAE system to state space form.

3.4.1 Reduced equation of motion

From the equilibrium condition of a constrained system one obtains the set of differential equations

$$[M]\,\{\ddot{q}\} + [\Psi_q]^T\,\{\lambda\} \;=\; \{g\} \tag{40}$$

It is convenient to convert $[\Psi_q]^T$ into $\left[\Psi''_q\right]^T$. Equation (15) can be rearranged in the form

$$\left[\Psi'_q\right]^T \;=\; [\Psi_q]^T\,[P]\,[Z]^T \quad\Rightarrow\quad [\Psi_q]^T \;=\; \left[\Psi'_q\right]^T [Z]\,[P]^T \tag{41}$$

Substituting (41) into (40) one obtains

$$[M]\,\{\ddot{q}\} + \left[\Psi'_q\right]^T [Z]\,[P]^T\,\{\lambda\} \;=\; \{g\} \tag{42}$$

Considering that

$$\left[\Psi'_q\right]^T \;=\; \left[\left[\Psi''_q\right]^T \;\; [0]\right] \tag{43}$$

and letting

$$\{\lambda'\} \;\equiv\; [Z]\,[P]^T\,\{\lambda\}^{6} \tag{44}$$

the expression

$$[M]\,\{\ddot{q}\} + \left[\left[\Psi''_q\right]^T \;\; [0]\right] \left\{ \begin{array}{c} \{\lambda'_1\} \\ \{\lambda'_2\} \end{array} \right\} \;=\; \{g\} \tag{45}$$

is obtained.

It is important to notice that all the $(m-r)$ elements of the vector $\{\lambda'_2\}$ are exactly zero, due to the trasformation made by $[Z]$, while $\{\lambda'_1\}$ is a vector formed by the first r elements of $\{\lambda\}$.

From equation (45), the reduced set of differential equations of motion can be obtained

$$[M]\,\{\ddot{q}\} + \left[\Psi''_q\right]^T \{\lambda'_1\} \;=\; \{g\}\ . \tag{46}$$

The most direct way to simulate the motion of the constrained system is by simultaneously solving the set (46) and the constraint equations. For this purpose, the use of the Fortran code `Radau` [5] is recommended for accurate results, with special regard to constraint violation.

3.4.2 Constraint orthogonalization

Taking into account the properties (34) and (35), the premultiplication of both members of equation (46) for $\left[\widehat{Q}\right]^T$ gives

$$\left[\begin{array}{c} \left[\widehat{Q}_1\right]^T \\ \left[\widehat{Q}_2\right]^T \end{array}\right] [M]\,\{\ddot{q}\} \;=\; \left[\begin{array}{c} \left[\widehat{Q}_1\right]^T \\ \left[\widehat{Q}_2\right]^T \end{array}\right] \{g\} - \left[\begin{array}{c} \left[\widehat{Q}_1\right]^T \\ \left[\widehat{Q}_2\right]^T \end{array}\right] \left[\Psi''_q\right]^T \{\lambda'_1\}$$

[6] Both $\{\lambda\}$ and $\{\lambda'\}$ are vectors with dimension m

$$\begin{bmatrix} \left[\widehat{Q}_1\right]^T [M]\{\ddot{q}\} \\ \left[\widehat{Q}_2\right]^T [M]\{\ddot{q}\} \end{bmatrix} = \begin{bmatrix} \left[\widehat{Q}_1\right]^T \{g\} \\ \left[\widehat{Q}_2\right]^T \{g\} \end{bmatrix} - \begin{bmatrix} \left[\widehat{Q}_1\right]^T \left[\Psi''_q\right]^T \\ \left[\widehat{Q}_2\right]^T \left[\Psi''_q\right]^T \end{bmatrix} \{\lambda'_1\}$$

$$\begin{cases} \left[\widehat{Q}_1\right]^T [M]\{\ddot{q}\} = \left[\widehat{Q}_1\right]^T \{g\} - \{\lambda'_1\} \\ \left[\widehat{Q}_2\right]^T [M]\{\ddot{q}\} = \left[\widehat{Q}_2\right]^T \{g\} \end{cases}$$

Finally, the following decoupled set of DAE is obtained

$$\begin{cases} \{\lambda'_1\} = \left[\widehat{Q}_1\right]^T (\{g\} - [M]\{\ddot{q}\}) \\ \left[\widehat{Q}_2\right]^T [M]\{\ddot{q}\} = \left[\widehat{Q}_2\right]^T \{g\} \end{cases} \tag{47}$$

3.4.3 State space representation with independent accelerations

Inserting accelerations relation (39) in the second equation of (47) gives

$$\left[\widehat{Q}_2\right]^T [M]\left[\widehat{Q}_1\right]\{a''\} + \left[\widehat{Q}_2\right]^T [M]\left[\widehat{Q}_2\right]\{\ddot{z}\} = \left[\widehat{Q}_2\right]^T \{g\} \tag{48}$$

From equations (38) and defining

$$\left[\widehat{M}\right] = \left[\widehat{Q}_2\right]^T [M]\left[\widehat{Q}_2\right] \tag{49}$$

$$\left[M'\right] = \left[\widehat{Q}_2\right]^T [M]\left[\widehat{Q}_1\right] \tag{50}$$

$$\left[M''\right] = \left[M'\right]\left[\bar{I}\right][Z][P]^T \tag{51}$$

equation (48) can be rewritten as

$$\left[\widehat{M}\right]\{\ddot{z}\} = \left[\widehat{Q}_2\right]^T \{g\} - \left[M'\right]\left[\bar{I}\right][Z][P]^T \left(-[\Psi_{qt}]\{\dot{q}\} - \{\Psi_{tt}\}\right) \tag{52}$$

Combining last equation with velocities relation (33), the following first order set of ordinary differential equations can be built

$$\begin{bmatrix} [I] & [0] \\ \left[-\left[M''\right][\Psi_{qt}]\right] & \left[\widehat{M}\right] \end{bmatrix} \begin{Bmatrix} \{\dot{q}\} \\ \{\ddot{z}\} \end{Bmatrix} = \begin{Bmatrix} \left[\widehat{Q}\right] \begin{Bmatrix} \{b''\} \\ \{\dot{z}\} \end{Bmatrix} \\ \left[M''\right]\{\Psi_{tt}\} + \left[\widehat{Q}_2\right]^T \{g\} \end{Bmatrix} \tag{53}$$

The coefficient matrix on the left–hand side of equation (53) is square, with dimensions $((2n - r) \times (2n - r))$ and well–conditioned.

The set of equations (53) does not represent globally the system, but it is only a local description. By the mean of this, after each computational step of the ODE solver, it is necessary to monitor and eventually correct the computed solution and to modify the very set of differential equations, in order to limit the amount of constraint violation below a specified value.

3.4.4 State space representation with a minimum set of coordinates

From the rows of the reduced Jacobian matrix $\left[\Psi_q''\right]$ one can search for a matrix $[B]$ such that [8]

$$\{\dot{p}\} = [B]\{\dot{q}\} \tag{54}$$

where the condition

$$\left[\begin{array}{c} \left[\Psi_q''\right] \\ [B] \end{array}\right]\{\dot{q}\} = \left\{\begin{array}{c} -\{\Psi_t''\} \\ \{\dot{p}\} \end{array}\right\} \tag{55}$$

must be fulfilled.

The solution of equation (55) can be written as

$$\begin{aligned} \{\dot{q}\} &= \left[\begin{array}{c} \left[\Psi_q''\right] \\ [B] \end{array}\right]^{-1}\left\{\begin{array}{c} -\{\Psi_t''\} \\ \{\dot{p}\} \end{array}\right\} = [[S]\ [V]]\left\{\begin{array}{c} -\{\Psi_t''\} \\ \{\dot{p}\} \end{array}\right\} \\ &= -[S]\{\Psi_t''\} + [V]\{\dot{p}\} \end{aligned} \tag{56}$$

Equation (56) yields

$$\begin{aligned} \left[\begin{array}{c} \left[\Psi_q''\right] \\ [B] \end{array}\right]\left[\begin{array}{c} \left[\Psi_q''\right] \\ [B] \end{array}\right]^{-1} &= \left[\begin{array}{c} \left[\Psi_q''\right] \\ [B] \end{array}\right][[S]\ [V]] = \\ &= \left[\begin{array}{cc} \left[\Psi_q''\right][S] & \left[\Psi_q''\right][V] \\ [B][S] & [B][V] \end{array}\right] = \left[\begin{array}{cc} [I] & [0] \\ [0] & [I] \end{array}\right] \end{aligned} \tag{57}$$

From properties (34) and (35), equation (57) gives [7]

$$[S] = \left[\widehat{Q}_1\right] \tag{58}$$

$$[V] = \left[\widehat{Q}_2\right] \tag{59}$$

$$[B] = \left[\widehat{Q}_2\right]^T \tag{60}$$

[7]Although $\left[\widehat{Q}_2\right]^T\left[\widehat{Q}_1\right] \neq 0$, if the Jacobian matrix is decomposed often, that is the time step is small enough, the difference between $[S]$ and $\left[\widehat{Q}_1\right]$ can be neglected from a numerical point of view. Moreover, using $\left[\widehat{Q}_1\right]$ instead of computing $\left[\begin{array}{c} \left[\Psi_q''\right] \\ [B] \end{array}\right]^{-1}$ and extracting from there the matrix $[S]$, avoids the calculation of the inverse of a $(n \times n)$ matrix, with benefits both for efficiency and for stability.

The time derivative of (55) gives the accelerations relation as

$$\left[\begin{array}{c}[\Psi''_q]\\ [B]\end{array}\right]\{\ddot{q}\} = \left\{\begin{array}{c}\{\gamma''\}\\ \{\ddot{p}\}\end{array}\right\} \tag{61}$$

where

$$\gamma'' = \left[\bar{I}\right][Z][P]^T\{\gamma\} \tag{62}$$

$$\{\gamma\} = -\left([\Psi_q]\{\dot{q}\}\right)_q\{\dot{q}\} - 2[\Psi_{qt}]\{\dot{q}\} - \{\Psi_{tt}\} \tag{63}$$

From the definitions of $[S]$ and $[V]$, equation (61) can be written as

$$\{\ddot{q}\} = [S]\{\gamma''\} + [V]\{\ddot{p}\} = \left[\widehat{Q}_1\right]\{\gamma''\} + \left[\widehat{Q}_2\right]\{\ddot{p}\} \tag{64}$$

The time integration of (55) gives the positions relation as

$$\{\Delta p\} = [B](\{\Delta q\} + \{\bar{q}\}) \tag{65}$$

where the constant vector is evaluated before each time step from the actual values of $\{q\}$.

Inserting (64) into the second equation of the set (47) gives

$$\left[\widehat{Q}_2\right]^T[M]\left[\widehat{Q}_2\right]\{\ddot{p}\} = \left[\widehat{Q}_2\right]^T\{g\} - \left[\widehat{Q}_2\right]^T[M]\left[\widehat{Q}_1\right]\{\gamma''\} \tag{66}$$

From the definitions (49) and (50), equation (66) can be written, in a more useful and compact form, as

$$\left[\widehat{M}\right]\{\ddot{p}\} = \left[\widehat{Q}_2\right]^T\{g\} - [M']\{\gamma''\} \tag{67}$$

This is the set of ordinary differential equations that is numerically integrated. In fact, with a standard ODE solver, in fact, it is possible to compute the vectors $\{p\}$, $\{\dot{p}\}$ and $\{\ddot{p}\}$ at the following time step. Then, using the relations (65), (54) and (61), generalized position, velocity and acceleration can be easily evaluated.

4 NUMERICAL EXAMPLES

All the following examples have been obtained using the general–purpose software `NumDyn3D` developed at University of Rome Tor Vergata [6, 7].

4.1 Parallel five–bar linkage

A parallel five–bar linkage is shown in Figure 1. Though the actual motion of the five–bar linkage is planar, the model includes links and revolute joints whose motion is defined in a three–dimensional space. Hence, the model has 7 redundant constraints and one degree-of-freedom.

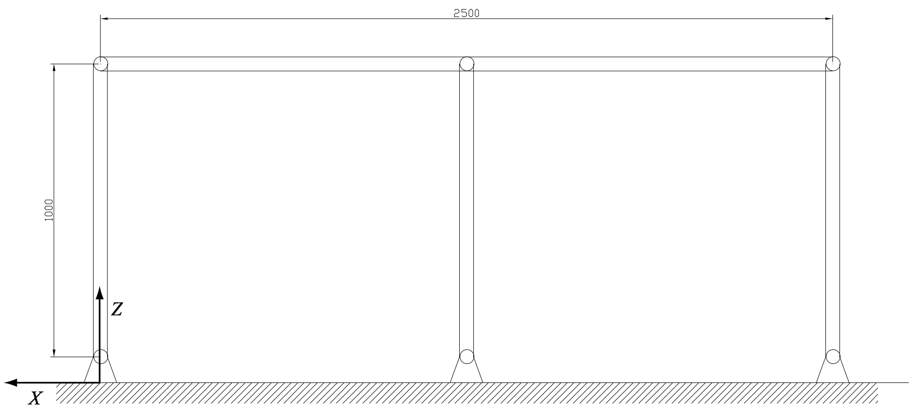

Figure 1: Parallel Five–Bar Linkage

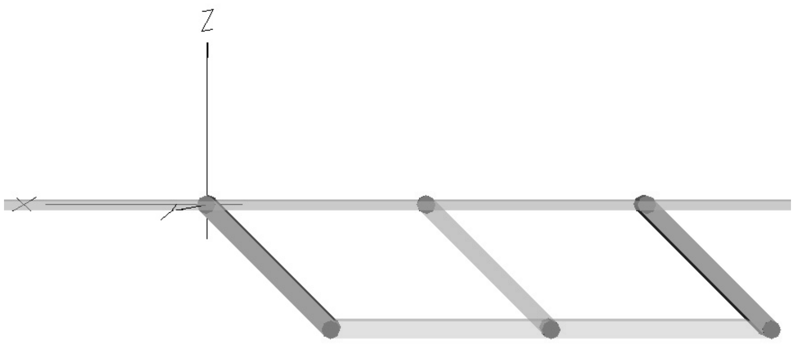

Figure 2: Five–Bar Linkage – Initial Configuration

The initial configuration is shown in Figure 2. The only external force is due to gravity and is directed downward. The system is free to oscillate. The results obtained solving directly, using Radau, equation (46) or reducing the algebraic–differential system of equations in state form, as in subsection 3.4.4 are compared to the results obtained with the commercial software Working Model.

The X coordinate of leftmost link is plotted versus time in figure 3, while constraint violation of one of the revolute joint connecting the same link to the frame is presented in figure 4.

Let one of the parallel bars have a constant angular velocity $\omega = 6.28\,\mathrm{rad/s}$.

The model has 7 dependent constraints and the only degree of freedom is driven by the driving constraint.

The computed position and velocity of the X coordinate of the crank are reported in figure 5, for a 20 s simulation, while in figure 6 are represented the computed trajectories of all the links centers of mass.

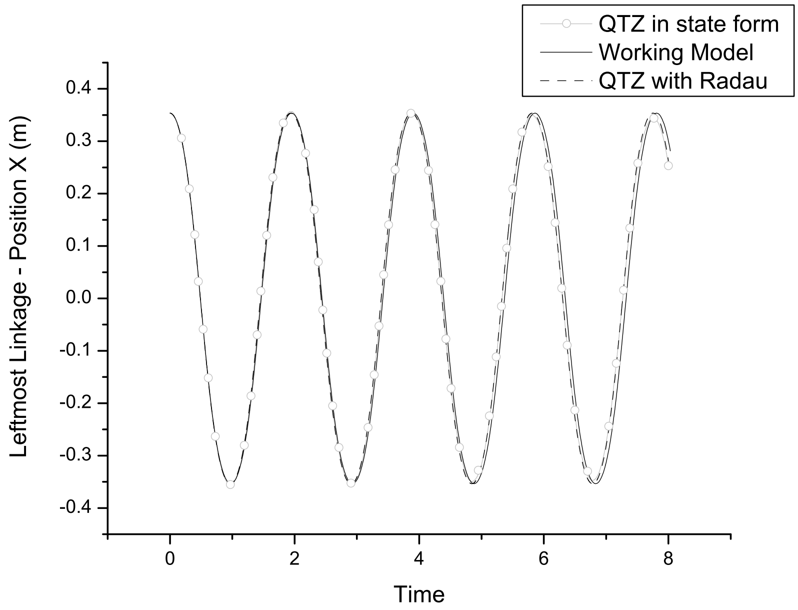

Figure 3: Five–Bar Linkage – Position of a Linkage

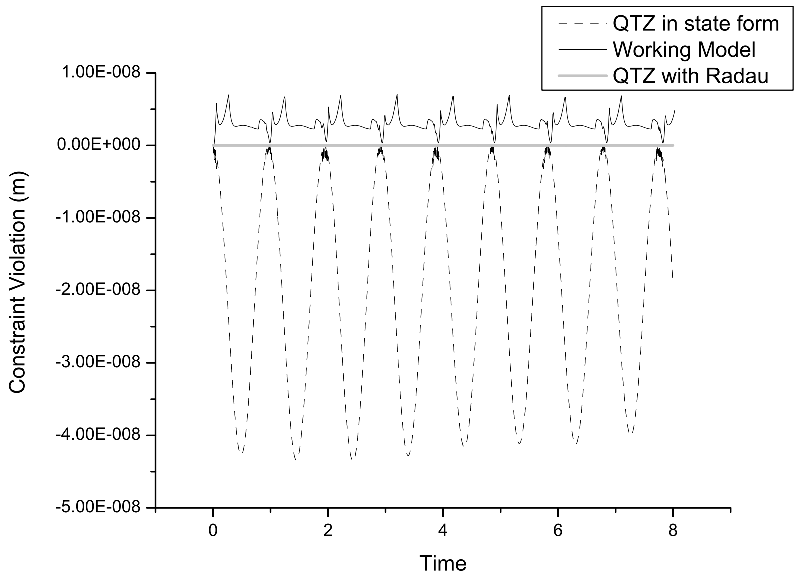

Figure 4: Five–Bar Linkage – Constraint Violation

The execution times for each simulation are summarized in table 1.

QTZ with Radau	4:07 min	5000 steps
QTZ in state form	3:13 min	5000 steps × 10 sub–steps
Working Model	6:08 min	5000 steps

Table 1: Computational time

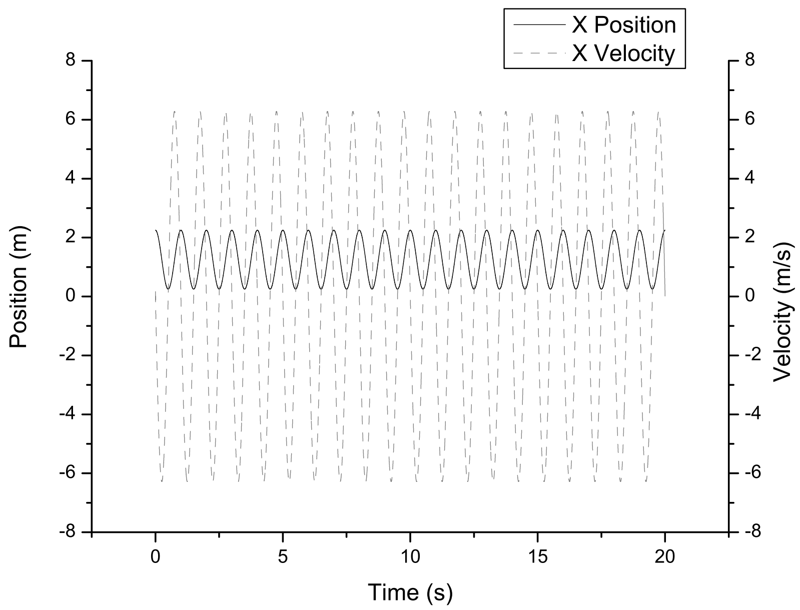

Figure 5: Five–Bar Linkage – Position and Velocity of Crank

4.2 Spherical pendulum with flywheel

Consider a straight rod connected to the frame through a spherical joint at one end and to a flywheel, through a revolute joint, at the other end. Let the axis of this joint pass through the center of the spherical joint and the center of the wheel. The system has four degrees-of-freedom and no redundant constraint and it is depicted in figure 7.

Masses and inertia matrices, in S.I. units and evaluated in local principal axes system of reference, are summarized in tables 2 and 3.

The only force present during the first 1.5 seconds is the gravity, directed along the negative Z axis, so that the pendulum is free to swing. A constant torque is then applicated to the wheel, directed along the z axis of its local system of reference, and it is removed after three more seconds, leaving the pendulm free to move under the influence of the gravity and the inertia.

The pendulum is espected to feel the effects of the gravity, that tend to let it swing, and of the gyroscopic effects introduced by the rotating wheel. This is confirmed by the results, obtained using the QTZ in state form formulation, shown in figures 8 and 9, in

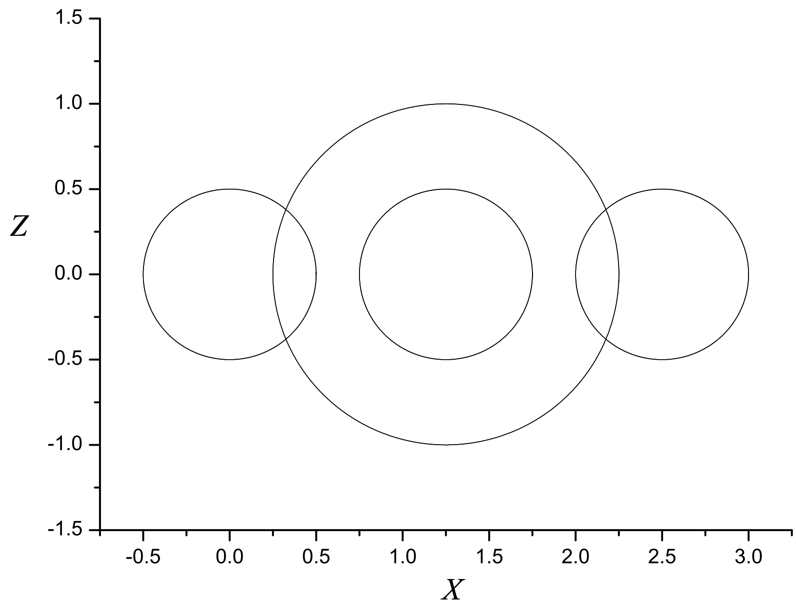

Figure 6: Five–Bar Linkage – Trajectories

Rod			
mass	2.71944		
Center of mass	0.	0.	0.
Inertia	0.05708	0.	0.
	0.	0.05708	0.
	0.	0.	0.0008498

Table 2: Inertia matrix of the rod

Wheel			
mass	4.89499		
Center of mass	0.	0.	0.
Inertia	0.02779	0.	0.
	0.	0.02779	0.
	0.	0.	0.05507

Table 3: Inertia matrix of the wheel

which the projections on the XY plane of the trajectory of the rod's center during two different phases of motion are shown. Initial free swing and the complete simulation are represented in a 3D space in figures 10 and 11.

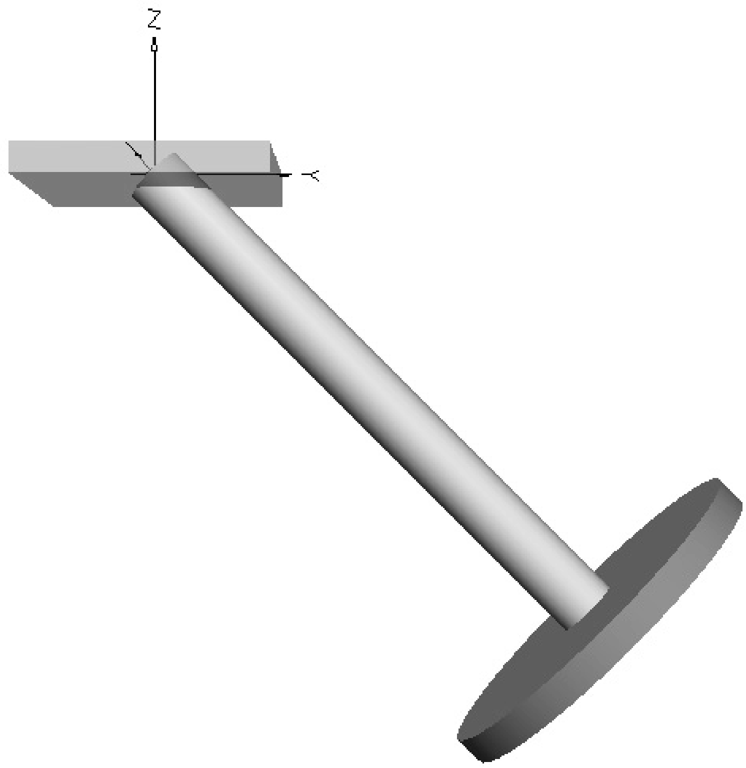

Figure 7: Spherical Pendulum with Flywheel – Initial Configuration

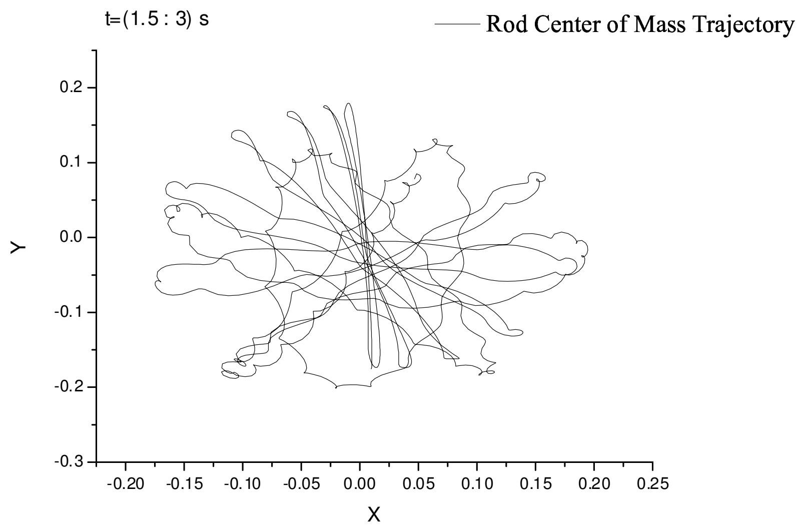

Figure 8: Spherical Pendulum with Flywheel – Center of the Wheel Trajectory

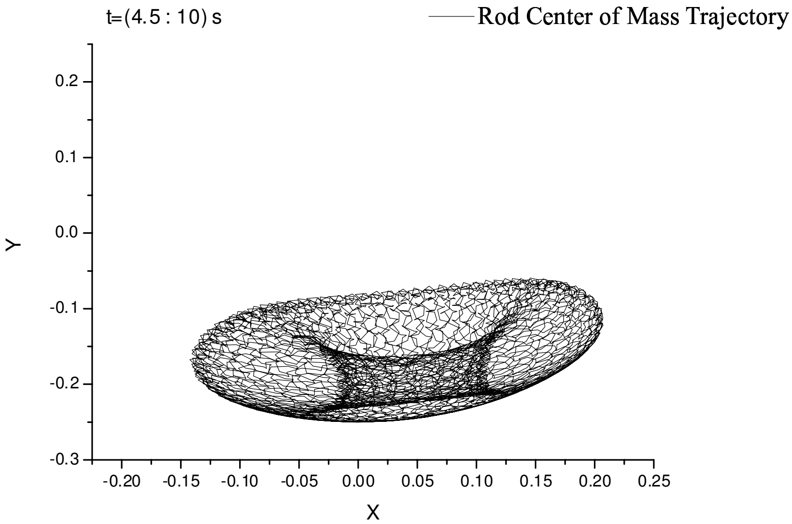

Figure 9: Spherical Pendulum with Flywheel – Free Motion

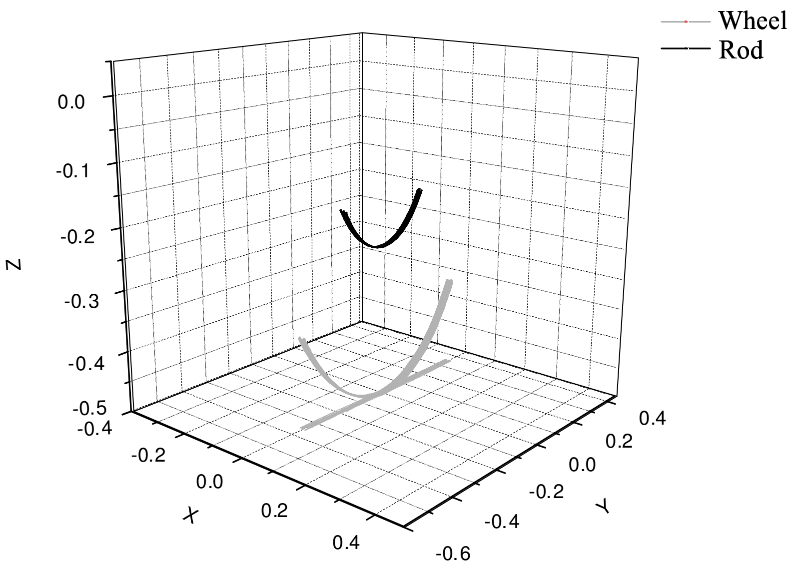

Figure 10: Spherical Pendulum with Flywheel – Free Initial Swing

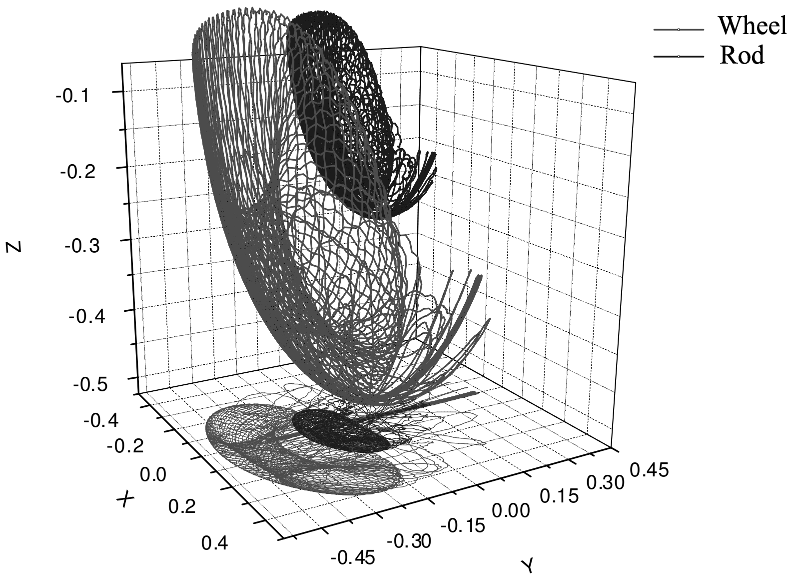

Figure 11: Spherical Pendulum with Flywheel – Whole Simulation

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5 APPENDIX

In this section is demonstrated how equation (30) satisfies both equation (28) and (29), and why the vector $\{\dot{z}\}$ has been defined as the vector of independent velocities.

Substituting the proposed solution (30) into the system (28), herein rewritten for a better comprehension

$$\left[\Psi_q''(\{q_0\}, t_0)\right]\{\dot{q}_0\} = \{b_0''\} ,$$

one obtains

$$\left[\Psi_q''(\{q_0\}, t_0)\right]\left\{\widehat{b_0''}\right\} + \left[\Psi_q''(\{q_0\}, t_0)\right][Q_2]\{\dot{z}\} = \{b_0''\} .$$

Considering that

$$\begin{aligned} \left\{\widehat{b}_0''\right\} &= [Q_1]\left([T]^T\right)^{-1}\{b_0''\} \ , \\ \left[\Psi_q''\right] &= [T]^T[Q_1]^T \ , \end{aligned}$$

the final expression is the following

$$\left[\Psi_q''\big(\{q_0\}, t_0\big)\right][Q_1]\left([T]^T\right)^{-1}\{b_0''\} + \left[\Psi_q''\big(\{q_0\}, t_0\big)\right][Q_2]\{\dot{z}\} = \{b_0''\} \ .$$

Since

$$\begin{aligned} [T]^T[Q_1]^T[Q_1]\left([T]^T\right)^{-1} &= [I] \ , \\ [Q_1]^T[Q_2] &= [0] \end{aligned}$$

the system (28) has been satisfied.

In the same manner, substituting the proposed solution (30) into the system (29), herein rewritten

$$[Q_1]^T\{\dot{q}_0\} = \left\{\widehat{b}_0''\right\}$$

one obtains

$$[Q_1]^T[Q_1]\left\{\hat{b}_0''\right\} + [Q_1]^T[Q_2]\{\dot{z}\} = \left\{\hat{b}_0''\right\} \ .$$

Considering the properties of matrix $[Q]$, previously discussed, also the system (29) has been satisfied.

Moreover it has been shown that the vector $\{\dot{z}\}$ does not take part directly to the solution of both systems, it is always multiplied by a null matrix. For this reason it has been defined as the vector of independent velocities freely prescribed by the user.